Math 402 - 01 Homework (Spring 2019)

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\inn}{\textrm{Inn}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\cl}{\textrm{cl}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \newcommand{\idl}{\textrm{Idl}} \newcommand{\lub}{\textrm{lub}} \newcommand{\glb}{\textrm{glb}} $

Problem Set 08 (partial) Due: 04/26/2019 Board presentation: 04/??/2019

1. Prove the following corollary to the Fundamental Theorem of Galois Theory. Use only the FTGT statements to prove it. Let E/F be a (finite) Galois extension, with Galois group $G=\gal_F(E)$. Let $L_1,L_2\in\sub_F(E)$ and $H_1,H_2\in\sub(G)$.
   I. $(L_1\meet L_2)^* = L_1^* \join L_2^*$
   II. $(L_1\join L_2)^* = L_1^* \meet L_2^*$
   III. $(H_1\meet H_2)^* = H_1^* \join H_2^*$
   IV. $(H_1\join H_2)^* = H_1^* \meet H_2^*$
2. Let $f(x)\in\Q[x]$ be such that it has a non-real root. Let E be the splitting field of $f(x)$ over $\Q$. Prove that $\gal_\Q(E)$ has even order.
3. Consider the polynomial $f(x)=x^3+2x^2+2x+2\in\Q[x]$, and E its splitting field over $\Q$.
   I. Show that $f(x)$ has exactly one real root. (Hint: use calculus)
   II. Show that $f(x)$ is irreducible over $\Q$.
   III. Find $[E:\Q]$. Fully explain your calculation.
   IV. Determine $\gal_\Q(E)$.

Problem Set 07 (complete) Due: 04/17/2019 Board presentation: 04/??/2019

1. Let E be a field, G a subgroup of $\aut(E)$, $F=E_G$, and $L\in\sub_F(E)$. Show that $L^*=\aut_L(E)$, and it is a subgroup of G.
2. Let E be a field, G a subgroup of $\aut(E)$, and $F=E_G$. Prove that for any $H,H_1,H_2\in\sub(G)$, and any $L,L_1,L_2\in\sub_F(E)$
   I. If $H_1 \leq H_2$, then $H_2^* \leq H_1^*$. (i.e. $\,^*$ is order reversing)
   II. If $L_1 \leq L_2$, then $L_2^* \leq L_1^*$. (i.e. $\,^*$ is order reversing)
   III. $H\leq H^{**}$ (i.e. $1 \leq \,^{**}$)
   IV. $L\leq L^{**}$ (i.e. $1 \leq \,^{**}$)

3. Let $E/L/F$ be a field tower.
    I. Prove that if E/F is a normal extension then so is E/L.
    II. Prove that if E/F is a Galois extension then so is E/L.

Problem Set 06 (complete) Due: 04/12/2019 Board presentation: 04/17/2019

1. Let F be a field, $\alpha_1,\dots,\alpha_n$ elements from some extension E of F, and R a commutative ring with unity. If $\varphi_1,\varphi_2:F(\alpha_1,\dots,\alpha_n)\to R$ are homomorphisms such that $\varphi_1(a)=\varphi_2(a)$ for all $a\in F$ and $\varphi_1(\alpha_i)=\varphi_2(\alpha_i)$ for $i=1,\dots,n$, then $\varphi_1=\varphi_2$.
2. Let $f(x)=x^5-2\in\Q[x]$, and E the splitting field of $f(x)$. Consider the group $G=\aut_\Q(E)$.
    I. What is the order of G?
    II. Is it abelian?
    III. What are the orders of elements in G?
3. Let $F=\F_p(t)$ be the field of rational functions on t with coefficients in $\F_p$. Consider the polynomial $f(x)=x^p-t\in F[x]$.
    I. Show that $f(x)$ has no root in F.
    II. Show that the Frobeni\us endomorphism $\Phi:F\to F$ is not surjective.
    III. Show that $f(x)$ has exactly one root, and that root has multiplicity p.
    IV. Show that $f(x)$ is irreducible over F.

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