# Math 401 - 01 Previous Homework (Fall 2018)

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\inn}{\textrm{Inn}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\cl}{\textrm{cl}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \newcommand{\idl}{\textrm{Idl}} \newcommand{\lub}{\textrm{lub}} \newcommand{\glb}{\textrm{glb}} $

**Problem Set 10** (complete) Due: 11/06/2018. Board presentation: 11/20/2018

1. Let $G$ be a group, and $H,K\leq G$.
   I. Prove that if $HK=KH$, then $HK\leq G$.
   II. Prove that if $H\leq N_G(K)$, then $HK\leq G$.
2. Let $G$ be a group, $H\leq G$, and $C=\{gHg^{-1}|g\in G\}$ the set of all conjugates of $H$ in $G$. Prove that: \[ |C|=[G:N_G(H)]. \]
3. Let $G$ be a group of order $120$. What are the possible values of $n_2$, $n_3$, and $n_5$, i.e. the number of Sylow 2-subgroups, the number of Sylow 3-subgroups and the number of Sylow 5-subgroups?
4. How many groups of order $6727$ are there? Describe them. Justify your answers. Show all your work.

**Problem Set 09** (complete) Due: 10/29/2018. Board presentation: 11/02/2018

1. Prove that, up to isomorphism, the direct product operation is commutative and associative.
2. Give an example of a group $G$ with two subgroups $H$ and $K$ such that $HK=G$, $H\intersection K=1$, $K\normaleq G$, but $G$ is not isomorphic to the direct product $H\oplus K$.
3. Let $G$ be a group, and $H,N\leq G$. Prove that:
   I. If $N\normaleq G$, then $HN\leq G$.
   II. If both $H,N\normaleq G$, then $HN\normaleq G$.
4. Make a list of all abelian groups of order $2736$. Express each of them using the *"elementary divisors"* form and the *"invariant factors"* form.

**Problem Set 08** (complete) Due: 10/22/2018. Problem 4 may be resubmitted by 10/24/2018. Board presentation 10/29/2018

1. Let $n\in\N$ and $H\leq G$ such that $H$ is the only subgroup of $G$ of order $n$. Show that $H\normaleq G$. (Do not assume that $G$ is finite)
2. Let $n\in\N$ and $H\leq G$ such that $H$ is the only subgroup of $G$ of index $n$. Show that $H\normaleq G$.

(Do not assume that $G$ is finite)

3. Combine the previous problem with problem 3 in Problem Set 6.
4. Let $p,q$ be primes such that $p < q$ and $p\not\mid (q-1)$. Prove that, up to isomorphism, there is only one group of order $pq$. (Hint: Use example 17, page 203, as a guide. No use this example, you may use the extra assumption that $(p-1)\not\mid (q-1)$, or equivalently that $(p-1)\not\mid (pq-1)$.)

**Problem Set 07** (complete) Due: 10/15/2018. Board presentation 10/29/2018

1. Prove Thm. 6.2.3, Thm. 6.3.2, Thm. 10.2.3. Combine all three proofs into one.
2. Chapter 10, problems 8, 10.

**Problem Set 06** (complete) Due: 10/08/2018. Board presentation 10/10/2018

1. Chapter 7, problem 8.
2. Chapter 7, problem 22.
3. Let $G$ be a finite group, and $p$ the smallest prime divisor of $|G|$. If $p^2\not\mid|G|$, then $G$ has at most one subgroup of index $p$. (Hint:Look at Example 6 on page 144)
4. Chapter 7, problem 12. Generalize.
5. Chapter 7, problem 48.

**Problem Set 05** (complete) Due: 09/24/2018. Board presentation 09/28/2018

1. Chapter 5, problems 6, 8. For all of them find the order and the parity.
2. Chapter 5, problem 10. What is the largest order of an element of $S_8$. Explain.
3. Chapter 5, problems 23, 24.
4. Chapter 5, problem 48.
5. Chapter 5, problem 50. Is $D_5$ a subgroup of $A_5$? Explain.

**Problem Set 04** (complete) Due: 09/17/2018. Board presentation: 09/24/2018

1. Chapter 4, problem 74.
2. Chapter 5, problem 2.a.
3. Chapter 5, problem 4.
4. Consider $\alpha\in S_8$ given in disjoint cycle form by $\alpha=(1\ 4\ 5)(3\ 7)$. Write $\alpha$ in array form.

**Problem Set 03** (complete) Due: 09/12/2018. Board presentation: 09/17/2018

1. Let $G=\pbr{a}$ be an infinite cyclic group, and $k_1,k_2\in\Z$. Prove that $$\pbr{a^{k_1}}\leq\pbr{a^{k_2}} \textrm{ iff } k_2\mid k_1.$$
2. Let $G=\pbr{a}$ be a cyclic group of order $60$.
   I. How many subgroups does $G$ have?
   II. Which of them are cyclic?
   III. List a generator for each of the cyclic subgroups of $G$.
   IV. Draw the subgroup lattice of $G$.
3. Prove that a finite group of prime order must be cyclic.
4. Chap. 4, problem 38, 62.
5. Chap. 4, problem 50.

**Problem Set 02** (complete) Due: 09/04/2018. Board presentation: 09/12/2018

1. Chap. 3, problems 4, 13, 20, 64
2. Chap. 3, problems 6, 50
3. Let $G$ be a group in which every non-identity element has order 2. Prove that $G$ must be Abelian.
4. Chap. 3, problem 34. what can you say about the union of two subgroups?

**Problem Set 01** (complete) Due: 08/27/2018. Board presentation: 08/31/2018

1. Page 38, prob. 18. What happens if you replace each H with an A?
2. Page 39, prob. 22. Explain.
3. Page 54, prob. 4.
4. Page 56, prob. 22. Compare with problem 13 on page 38.

Additional problems to look at: 1.13 p.38, 1.14 p.38, 2.5 p.54,

Home