Technische Universität Braunschweig

Calculating the Frattini subgroup of a polycyclic group

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Introduction

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- Gaschütz [1] and Eick [2], based on pre-Frattini groups (does not generalise to the infinite case).
- Baumslag, Cannonito, Robinson and Segal [3] (not practical)

Polycyclic groups: Definition and Examples

Definition: (Hirsch 1938 [4])

A group G is polycyclic if there exists a series of subnormal subgroups

$$G = G_1 > G_2 > \ldots > G_n > G_{n+1} = \{1\}$$

so that G_i / G_{i+1} is (finite or infinite) cyclic for $1 \leqslant i \leqslant n$.

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Example:

The following groups are polycyclic:

- *finite soluble groups*
- *finitely generated abelian or nilpotent groups*
- *wallpaper groups, space groups*
- *extensions of polycyclic groups by polycyclic groups*

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so that G_i / G_{i+1} *is (finite or infinite) cyclic for* $1 \leqslant i \leqslant n$.

Example:

*The following groups are **not** polycyclic:*

- *groups with non-abelian, free subgroups*
- *groups with non-finitely generated subgroups*
- *non-soluble groups*

The algorithm: basic idea

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\end{aligned}$$

so if $\phi(G/N)$ and $I(G, N)$ can be calculated effectively, an effective calculation of $\phi(G)$ is possible.

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- Let $F = Fit(G)$. If G does not act semisimply on F/F^p for all but finitely many primes p, then there is a normal subgroup $L \trianglelefteq G$ with $L \leqslant \phi(G)$ that can be calculated from the radical of $F \otimes_{\mathbb{Z}} \mathbb{Q}$. Choose $N = L$.

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- If none of the above approaches yield nontrivial normal subgroups, $\phi(G)$ is trivial.

The algorithm: infinite cyclic central normal subgroups

Definition:

Let G be polycyclic, $N \cong \mathbb{Z}$ and $N \leqslant Z(G)$. Then $I_p(G, N)$ denotes the intersection of all maximal subgroups M of G having p-power index in G and satisfying $N \not\leqslant M$.

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Theorem:

Let N be an infinite cyclic, central subgroup of G. Then one of the following cases applies:

1. *If N/N^p only has complements in G/N^p for finitely many p, then there exists a finite set of primes P so that $I(G, N) = \cap_{p \in P} I_p(G, N)$.*
2. *If N/N^p has a complement in G/N^p for infinitely many primes p, then $I(G, N) = G'$.*

The algorithm: Semisimplicity

Theorem:

Let G be polycyclic, $N \trianglelefteq G$, $N \cong \mathbb{Z}^d$. Then one of the following two cases holds:

1. *G acts semisimply on N/N^p for almost all primes p. This is the case if and only if the induced action of G on $\mathbb{Q}^d = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple.*
2. *G acts semisimply on N/N^p for only finitely primes p. This is the case if and only if the induced action of G on $\mathbb{Q}^d = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is not semisimple.*

The algorithm: the remaining case

Definition:

Let G be polycyclic and $F = Fit(G)$. If

- *F is free abelian and*
- *G has no finite non-trivial normal subgroups,*
- *G has trivial centre,*
- *G acts semisimply on $F \otimes_{\mathbb{Z}} \mathbb{Q}$*

then G has property $\mathcal{P}$.

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then G has property $\mathcal{P}$.

The normal subgroups discussed so far yield an iteration step if and only if G does not have property $\mathcal{P}$.

The algorithm: the remaining case

Theorem:

Let G have property $\mathcal{P}$ and $F = Fit(G)$. Then:

- $H^2(G/F, F)$ *is finite,*
- F/F^p *has a complement in* G/F^p *for almost all primes* p *and*
- $\phi(G)$ *is trivial.*

The algorithm: the remaining case

Theorem:

Let G have property $\mathcal{P}$ and $F = Fit(G)$. Then:

- $H^2(G/F, F)$ *is finite,*
- F/F^p *has a complement in* G/F^p *for almost all primes* p *and*
- $\phi(G)$ *is trivial.*

The algorithm yields an answer after a finite number of iterations, because

- polycyclic groups are Hopfian and
- every ascending series of (normal) subgroups stabilizes after finitely many steps.

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