On the computability of the abelian kernel of an inverse semigroup

Vicent Pérez Calabuig

Universitat de València

2021 Zassenhaus Groups and Friends Conference
28th May of 2021

1. Background
2. Computing the abelian kernel of an inverse semigroup
3. Applications: the problem of computing proabelian closures

1. Background

2. Computing the abelian kernel of an inverse semigroup

3. Applications: the problem of computing proabelian closures

Background: Rhodes' Type II Conjecture

Kernel of a semigroup S: The set of all elements that relate to 1 under every relational morphism from S to a group.

Rhodes and Tilson's seminal paper (1972): A lower bound for the complexity of a semigroup by means of kernels of semigroup.

The kernel of a regular semigroup is its smallest subsemigroup closed under weak conjugation.

- J. Rhodes and B. R. Tilson, *Improved lower bounds for the complexity of finite semigroups*, J. Pure Appl. Algebra 2 (1972) 13–71.

Kernel of a semigroup S: The set of all elements that relate to 1 under every relational morphism from S to a group.

Rhodes and Tilson's seminal paper (1972): A lower bound for the complexity of a semigroup by means of kernels of semigroup.

> The kernel of a regular semigroup is its smallest subsemigroup closed under weak conjugation.

- J. Rhodes and B. R. Tilson, *Improved lower bounds for the complexity of finite semigroups*, J. Pure Appl. Algebra **2** (1972) 13–71.

Rhodes' Type II Conjecture

Strong form: The kernel of every semigroup is the smallest subsemigroup closed under weak conjugation.

Weak form: The kernel of every semigroup is computable

Background: Type II Theorem

Ash's proof (1991): Structural proof using inevitable graphs where inverse semigroups play an important role.

Ribes and Zaleskiĭ's proof (1993): Translation of the problem to profinite topology by Pin and Reutenauer.

- C. J. Ash, *Inevitable graphs: a proof of the type II conjecture and some related decision procedures*, Int. J. Algebra Comput., 1 (1991), 127–146.
- L. Ribes and P. Zaleskiĭ, *On the profinite topology on a free group*, Bull. London Math. Soc., 25 (1993), 37–43.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathrm{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathrm{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} \textcircled{m} \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

The $\mathfrak{F}$-kernel of a semigroup S, $\mathrm{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathrm{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} \textcircled{m} \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathrm{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathrm{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} \textcircled{m} \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathrm{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathrm{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} ⓜ \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathsf{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathsf{K}_{\mathfrak{F}}(S) = \mathsf{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathsf{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} \textcircled{m} \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathsf{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathsf{K}_{\mathfrak{F}}(S) = \mathsf{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathsf{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} \textcircled{m} \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Generalised kernels

The $\mathfrak{F}$-kernel of a semigroup S, $\mathrm{K}_{\mathfrak{F}}(S)$: The set of all elements that relate to 1 under every relational morphism from S to a group $G \in \mathfrak{F}$.

$\mathfrak{F} = \mathfrak{G}$, then $\mathrm{K}_{\mathfrak{F}}(S) = \mathrm{K}(S)$.

The generalised Rhodes' Type II Conjecture: Given a variety of groups $\mathfrak{F}$, $\mathrm{K}_{\mathfrak{F}}(S)$ is computable.

- The computability of Mal'cev products of the form $\mathbf{V} ⓜ \mathfrak{F}$.
- For $\mathfrak{F}$ extension-closed: the computability of the closure of a finitely generated subgroup in the pro-$\mathfrak{F}$ topology of a free group.
- $\mathfrak{F}$: soluble groups or groups of odd order. Model Theory and logical descriptions of formal languages.

Background: Partial results

- Variety of all p-groups (Ribes and Zalesskiĭ, 1994).
  - L. Ribes and P. Zaleskiĭ, *The pro-p topology of a free group and algorithmic problems in semigroups*, Int. J. Algebra Comput., 4 (1994), 359–374.
- Variety of all abelian groups (Delgado, 1998) or every variety of abelian groups with decidable membership (Steinberg, 1999)
  - M. Delgado, *Abelian pointlikes of a monoid*, Semigroup Forum 56 (1998), 339–361.
  - B. Steinberg, *Monoid kernels and profinite topologies on the free abelian group*, Bull. Austral. Math. Soc., 60 (1999), 391–402.

Background: Partial results

- Variety of all p-groups (Ribes and Zalesskiĭ, 1994).
  - L. Ribes and P. Zaleskiĭ, *The pro-p topology of a free group and algorithmic problems in semigroups*, Int. J. Algebra Comput., 4 (1994), 359–374.
- Variety of all abelian groups (Delgado, 1998) or every variety of abelian groups with decidable membership (Steinberg, 1999)
  - M. Delgado, *Abelian pointlikes of a monoid*, Semigroup Forum 56 (1998), 339–361.
  - B. Steinberg, *Monoid kernels and profinite topologies on the free abelian group*, Bull. Austral. Math. Soc., 60 (1999), 391–402.

- Variety of all nilpotent groups (Almeida, Shahzamanian and Steinberg, 2017)
    - J. Ameida and H. Shahzamanian and B.Steinberg, *The pro-nilpotent group topology on a free group*, J. Algebra, 480 (2017), 332–345.

Variety of all abelian groups (Delgado, 1998) or every variety of abelian groups with decidable membership (Steinberg, 1999)

- M. Delgado, *Abelian pointlikes of a monoid*, Semigroup Forum 56 (1998), 339–361.
- B. Steinberg, *Monoid kernels and profinite topologies on the free abelian group*, Bull. Austral. Math. Soc., 60 (1999), 391–402.

Basis of the proof

Reduce the calculation of the abelian kernel to the calculation of the closure of certain subsets of the free commutative monoid and then test if 0 belongs to this closure.

Background: Abelian case

Our approach consists on:

- Characterise how relational morphisms between inverse semigroups and abelian groups are.
- Describe the elements of a $\mathcal{J}$-class of an inverse semigroups which are in the abelian kernel.
- In addition, a family of finite abelian groups which provide the abelian kernel is also described.

Our approach consists on:

- Characterise how relational morphisms between inverse semigroups and abelian groups are.
- Describe the elements of a $\mathcal{J}$-class of an inverse semigroups which are in the abelian kernel.
- In addition, a family of finite abelian groups which provide the abelian kernel is also described.

Our approach consists on:

- Characterise how relational morphisms between inverse semigroups and abelian groups are.
- Describe the elements of a $\mathcal{J}$-class of an inverse semigroups which are in the abelian kernel.
- In addition, a family of finite abelian groups which provide the abelian kernel is also described.

The abelian kernel of an inverse semigroup

Abelian kernel: Starting point

Let $\mathfrak{F}$ be a variety of groups and S an inverse semigroup.

$\mathrm{K}_{\mathfrak{F}}(S)$ is computable if, and only if, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S.

Abelian kernel: Starting point

Let $\mathfrak{F}$ be a variety of groups and S an inverse semigroup.

$\mathrm{K}_{\mathfrak{F}}(S)$ is computable if, and only if, $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S.

Given $J = J_s$ a $\mathcal{J}$-class of S, take

$$X_J = \{t \in S : s \leq_{\mathcal{J}} t\}$$

$S \setminus X_J$ is an ideal of S. Then, $S_J := S/(S \setminus X_J)$ is an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class, J.

$$\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(S_J) \cap J$$

As a consequence:

$\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S if, and only if, $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable, for every inverse semigroup $\bar{S}$ with a unique 0-minimal $\mathcal{J}$-class, $\bar{J}$.

Given $J = J_s$ a $\mathcal{J}$-class of S, take

$$X_J = \{t \in S : s \leq_{\mathcal{J}} t\}$$

$S \setminus X_J$ is an ideal of S. Then, $S_J := S/(S \setminus X_J)$ is an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class, J.

$$\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(S_J) \cap J$$

As a consequence:

$\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S if, and only if, $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable, for every inverse semigroup $\bar{S}$ with a unique 0-minimal $\mathcal{J}$-class, $\bar{J}$.

Given $J = J_s$ a $\mathcal{J}$-class of S, take

$$X_J = \{t \in S : s \leq_{\mathcal{J}} t\}$$

$S \setminus X_J$ is an ideal of S. Then, $S_J := S/(S \setminus X_J)$ is an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class, J.

$$\mathrm{K}_{\mathfrak{F}}(S) \cap J = \mathrm{K}_{\mathfrak{F}}(S_J) \cap J$$

As a consequence:

$\mathrm{K}_{\mathfrak{F}}(S) \cap J$ is computable, for every $\mathcal{J}$-class J of S if, and only if, $\mathrm{K}_{\mathfrak{F}}(\bar{S}) \cap \bar{J}$ is computable, for every inverse semigroup $\bar{S}$ with a unique 0-minimal $\mathcal{J}$-class, $\bar{J}$.

Abelian kernel: Minimal pairs

- Let S be an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class J. Then, J^0 is a Brandt semigroup.
- We can consider $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, with G a group and Λ a set of indices

$$J^0 = \{(i, g, j) : i, j \in \Lambda,\ g \in G\} \cup \{0\}$$

with the product

$$(i, g, j) \cdot (i', g', j') = \begin{cases} (i, gg', j'), & j = i' \\ 0, & j' \neq i \end{cases}$$

Abelian kernel: Minimal pairs

- Let S be an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class J. Then, J^0 is a Brandt semigroup.

- We can consider $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, with G a group and Λ a set of indices

$$J^0 = \{(i, g, j) : i, j \in \Lambda,\ g \in G\} \cup \{0\}$$

with the product

$$(i, g, j) \cdot (i', g', j') = \begin{cases} (i, gg', j'), & j = i' \\ 0, & j' \neq i \end{cases}$$

Abelian kernel: Minimal pairs

- Let $\mathcal{U}_\Lambda(G)$ be the inverse semigroup of $\Lambda \times \Lambda$ matrices which have at most one non-zero entry in G for each row and column. It holds $\mathcal{U}_\Lambda(G)$ has a unique 0-minimal $\mathcal{J}$-class

$$\bar{J} = \{M_{(i,g,j)} : i,j \in \Lambda,\ g \in G\}$$

  such that J^0 and $\bar{J}^0$ are isomorphic.
- Then, we define a homomorphism $\phi_S \colon S \to \mathcal{U}_\Lambda(G)$. For every $s \in S$, take $M_s = \varphi_S(s)$. Whenever $M_s(i,j) = g$, we say that (i,g,j) is a *projection of* s *onto* J. We call M_s the *matrix of projections* of s onto J.

Abelian kernel: Reduction theorems

Definition

Let S be an inverse semigroup with a 0-minimal $\mathcal{J}$-class J isomorphic to $\mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. We say that (S, J) is a *minimal pair* if φ_S is a monomorphism, i.e. for every $s, t \in S$, $M_s = M_t$ implies $s = t$.

Proposition

The computation of $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ can be reduced to the case of minimal pairs (S, J).

Theorem

The $\mathfrak{F}$-kernel of every inverse semigroup is computable if, and only if, the $\mathfrak{F}$-kernel of every minimal pair with all maximal subgroups in $\mathfrak{F}$ is computable.

Abelian kernel: Reduction theorems

Definition

Let S be an inverse semigroup with a 0-minimal $\mathcal{J}$-class J isomorphic to $\mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. We say that (S, J) is a *minimal pair* if φ_S is a monomorphism, i.e. for every $s, t \in S$, $M_s = M_t$ implies $s = t$.

Proposition

The computation of $\mathrm{K}_{\mathfrak{F}}(S) \cap J$ can be reduced to the case of minimal pairs (S, J).

Theorem

The $\mathfrak{F}$-kernel of every inverse semigroup is computable if, and only if, the $\mathfrak{F}$-kernel of every minimal pair with all maximal subgroups in $\mathfrak{F}$ is computable.

As a consequence: *S is an inverse semigroup with a unique 0-minimal $\mathcal{J}$-class J, such that every maximal subgroup in S is abelian.*

In particular, we can assume $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, where G is an abelian group.

As a consequence: *S is an inverse semigroup with a unique* 0-*minimal $\mathcal{J}$-class J, such that every maximal subgroup in S is abelian.*

In particular, we can assume $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$, where G is an abelian group.

Key Lemma

Let $\tau: S \multimap A$, with $A \in \mathfrak{Ab}$. Then, the following statements hold:

1. *For every $i, j \in \Lambda$, $\tau(i,1,i) = \tau(j,1,j) =: H \trianglelefteq A$. Then, for every $(i,g,j) \in J$, $\tau(i,g,j) = x + H$, for each $x \in \tau(i,g,j)$.*
2. *For every $0 \neq s \in S$, the equality $\tau(i,g,j) = \tau(i',g',j')$ holds for all $M_s(i,j) = g \neq 0$, $M_s(i',j') = g' \neq 0$.*
3. *There exists $\bar{\tau}: S \multimap A/H$ such that $|\bar{\tau}(s)| = 1$, for every $0 \neq s$, and $\bar{\tau}^{-1}(H) \cap J = \tau^{-1}(0) \cap J$.*

Key Lemma

Let $\tau\colon\ S \longrightarrow\!\!\!\circ\!\!\!\rightarrow A$, with $A \in \mathfrak{Ab}$. Then, the following statements hold:

1. *For every $i, j \in \Lambda$, $\tau(i,1,i) = \tau(j,1,j) =: H \trianglelefteq A$. Then, for every $(i,g,j) \in J$, $\tau(i,g,j) = x + H$, for each $x \in \tau(i,g,j)$.*
2. *For every $0 \neq s \in S$, the equality $\tau(i,g,j) = \tau(i',g',j')$ holds for all $M_s(i,j) = g \neq 0$, $M_s(i',j') = g' \neq 0$.*
3. *There exists $\overline{\tau}\colon\ S \longrightarrow\!\!\!\circ\!\!\!\rightarrow A/H$ such that $|\overline{\tau}(s)| = 1$, for every $0 \neq s$, and $\overline{\tau}^{-1}(H) \cap J = \tau^{-1}(0) \cap J$.*

Key Lemma

Let $\tau\colon\ S \multimap A$, with $A \in \mathfrak{Ab}$. Then, the following statements hold:

1. *For every $i, j \in \Lambda$, $\tau(i, 1, i) = \tau(j, 1, j) =: H \trianglelefteq A$. Then, for every $(i, g, j) \in J$, $\tau(i, g, j) = x + H$, for each $x \in \tau(i, g, j)$.*
2. *For every $0 \neq s \in S$, the equality $\tau(i, g, j) = \tau(i', g', j')$ holds for all $M_s(i, j) = g \neq 0$, $M_s(i', j') = g' \neq 0$.*
3. *There exists $\bar{\tau}\colon\ S \multimap A/H$ such that $|\bar{\tau}(s)| = 1$, for every $0 \neq s$, and $\bar{\tau}^{-1}(H) \cap J = \tau^{-1}(0) \cap J$.*

Abelian kernel: Relational morphisms

Key Lemma

Let $\tau\colon\ S \multimap A$, with $A \in \mathfrak{Ab}$. Then, the following statements hold:

1. *For every $i, j \in \Lambda$, $\tau(i,1,i) = \tau(j,1,j) =: H \trianglelefteq A$. Then, for every $(i,g,j) \in J$, $\tau(i,g,j) = x + H$, for each $x \in \tau(i,g,j)$.*
2. *For every $0 \neq s \in S$, the equality $\tau(i,g,j) = \tau(i',g',j')$ holds for all $M_s(i,j) = g \neq 0$, $M_s(i',j') = g' \neq 0$.*
3. *There exists $\bar{\tau}\colon\ S \multimap A/H$ such that $|\bar{\tau}(s)| = 1$, for every $0 \neq s$, and $\bar{\tau}^{-1}(H) \cap J = \tau^{-1}(0) \cap J$.*

Consequences

For every minimal pair (S, J), there exists a relational morphism $\tau\colon\ S \multimap A \in \mathfrak{Ab}$ such that:

- $\tau^{-1}(0) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S) \cap J$.
- for every $0 \neq s \in S$, $|\tau(s)| = 1$ and $\tau(s) = \tau(i, g, j)$, for each non-zero entry $M_s(i, j) = g$ of the matrix of projections M_s.
- for every $s, t \in S$ with $0 \neq st$ the equality $\tau(st) = \tau(s) + \tau(t)$ holds.

Abelian kernel: Main Theorem

Theorem

Let (S, J) be a minimal pair with $J^0 = \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $A := F_{ab,\Lambda} \oplus G$ be the infinite abelian group, where $F_{ab,\Lambda}$ is the free abelian group over the alphabet Λ. Then

$$\mathrm{K}_{\mathfrak{Ab}}(S) \cap J = \{(i, g, j) : (-i + j, g) \in N\},$$

where $N = \langle N_s : 0 \neq s \in S \rangle \trianglelefteq F_{ab,X} \oplus G$ *with*

$$N_s := \{(-i + j, g_1) - (-i' + j', g_2) : M_s(i, j) = g_1,\ M_s(i', j') = g_2\},$$

for every $0 \neq s \in S$.

Remarks of the proof

1. The proof is constructive.
2. Although the quotient A/N could be not finite, in the first part of the proof, we see that every punctual relational morphism $\tau : S \multimap B$, which provides the abelian kernel, satisfies that B is isomorphic to a quotient of A/N.
3. In the second part of the proof we construct a family of abelian finite groups, which are quotient of A/N, and such that we can construct a relational morphism providing the abelian kernel.

4. In particular, depending on S it is possible to define a normal subgroup $\bar{N}$ such that $N \leq \bar{N} \trianglelefteq A$, $A/\bar{N}$ is finite and we construct $\tau\colon\; S \multimap A/\bar{N}$ satisfying

$$\tau^{-1}(0) \cap J = \mathrm{K}_{\mathfrak{Ab}}(S) \cap J = \{(i, g, j) : (-i + j, g) \in N\}$$

Applications

- Finitely generated subgroups of a free group F_X over an alphabet X and the proabelian topology

    1. Problem of extending partial injective maps into permutations that generate an abelian group (equivalent to compute the least extendible finitely generated subgroup)

    2. Problem of computing the proabelian closure

- Finitely generated subgroups of a free group F_X over an alphabet X and the proabelian topology

1. Problem of extending partial injective maps into permutations that generate an abelian group (equivalent to compute the least extendible finitely generated subgroup)
2. Problem of computing the proabelian closure

- Finitely generated subgroups of a free group F_X over an alphabet X and the proabelian topology

1. Problem of extending partial injective maps into permutations that generate an abelian group (equivalent to compute the least extendible finitely generated subgroup)

2. Problem of computing the proabelian closure

Let X be an alphabet and let H be a finitely generated subgroup of the free group F_X.

- There exists an inverse automaton associated with H, $\mathcal{A}(H)$.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ given by the transition morphism $\mu\colon \tilde{X}^* \to M(H)$, where $\tilde{X}^*$ is the free inverse monoid over X.
- $M(H)$ is the inverse monoid of the partial injective maps over the vertices of $\mathcal{A}(H)$.

Proposition

$M(H)$ has a unique 0-minimal $\mathcal{J}$-class J such that $(M(H), J)$ is a minimal pair.

Let X be an alphabet and let H be a finitely generated subgroup of the free group F_X.

- There exists an inverse automaton associated with H, $\mathcal{A}(H)$.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ given by the transition morphism $\mu \colon \tilde{X}^* \to M(H)$, where $\tilde{X}^*$ is the free inverse monoid over X.
- $M(H)$ is the inverse monoid of the partial injective maps over the vertices of $\mathcal{A}(H)$.

Proposition

$M(H)$ has a unique 0-minimal $\mathcal{J}$-class J such that $(M(H), J)$ is a minimal pair.

Let X be an alphabet and let H be a finitely generated subgroup of the free group F_X.

- There exists an inverse automaton associated with H, $\mathcal{A}(H)$.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ given by the transition morphism $\mu\colon \tilde{X}^* \to M(H)$, where $\tilde{X}^*$ is the free inverse monoid over X.
- $M(H)$ is the inverse monoid of the partial injective maps over the vertices of $\mathcal{A}(H)$.

Proposition

$M(H)$ has a unique 0-minimal $\mathcal{J}$-class J such that $(M(H), J)$ is a minimal pair.

Let X be an alphabet and let H be a finitely generated subgroup of the free group F_X.

- There exists an inverse automaton associated with H, $\mathcal{A}(H)$.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ given by the transition morphism $\mu\colon \tilde{X}^* \to M(H)$, where $\tilde{X}^*$ is the free inverse monoid over X.
- $M(H)$ is the inverse monoid of the partial injective maps over the vertices of $\mathcal{A}(H)$.

Proposition

$M(H)$ has a unique 0-minimal $\mathcal{J}$-class J such that $(M(H), J)$ is a minimal pair.

Call $\sim$ the least automaton congruence such that $\mathcal{A}(H)/\sim$ is extendible into a complete inverse automaton.

Proposition

The automaton congruence is given by $v \sim v'$ if, and only if, there exists $m \in \mathrm{K}_{\mathfrak{Ab}}(M(H)) \cap J$ such that $v \cdot m = v'$.

Solution of Problem 1 is given by our main theorem. Moreover, it also gives an abelian group which can extend such partial injective maps.

Call $\sim$ the least automaton congruence such that $\mathcal{A}(H)/\sim$ is extendible into a complete inverse automaton.

Proposition

The automaton congruence is given by $v \sim v'$ if, and only if, there exists $m \in \mathrm{K}_{\mathfrak{Ab}}(M(H)) \cap J$ such that $v \cdot m = v'$.

Solution of Problem 1 is given by our main theorem. Moreover, it also gives an abelian group which can extend such partial injective maps.

2. Problem of computing the proabelian closure of a finitely generated subgroup of a free group

- Let H be a finitely generated subgroup of a free group F_X over the alphabet X
- Let $\mathcal{A}(H)$ be the inverse automaton associated with H.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ with a unique 0-minimal $\mathcal{J}$-class, so that $(M(H), J)$ is a minimal pair and $\mu\colon \tilde{X}^* \to M(H)$ the transition morphism.

2. Problem of computing the proabelian closure of a finitely generated subgroup of a free group

- Let H be a finitely generated subgroup of a free group F_X over the alphabet X
- Let $\mathcal{A}(H)$ be the inverse automaton associated with H.
- Let $M(H)$ be the transition monoid of $\mathcal{A}(H)$ with a unique 0-minimal $\mathcal{J}$-class, so that $(M(H), J)$ is a minimal pair and $\mu \colon \tilde{X}^* \to M(H)$ the transition morphism.

- Let $A = F_{ab,\Lambda} \oplus G/G'$, where $J^0 \cong \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $N = \langle N_s : s \in M(H) \rangle \trianglelefteq A$, where

$$N_s = \{(-i+j, gG') - (-i'+j', g'G') : M_s(i,j) = g\ M_s(i',j') = g'\}$$

- We are able to construct $\bar{f} : F_X \to A/N$ which gives the following result.

Theorem

The proabelian closure of H is $\ker \bar{f}$. Moreover, $\mathrm{Cl}(H)$ is finitely generated if, and only if, A/N is finite.

- Let $A = F_{ab,\Lambda} \oplus G/G'$, where $J^0 \cong \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $N = \langle N_s : s \in M(H) \rangle \trianglelefteq A$, where

$$N_s = \{(-i+j, gG') - (-i'+j', g'G') : M_s(i,j) = g \; M_s(i',j') = g'\}$$

- We are able to construct $\overline{f} \colon F_X \to A/N$ which gives the following result.

Theorem

The proabelian closure of H is $\ker \overline{f}$. Moreover, $\mathrm{Cl}(H)$ is finitely generated if, and only if, A/N is finite.

- Let $A = F_{ab,\Lambda} \oplus G/G'$, where $J^0 \cong \mathcal{M}^0(G, \Lambda, \Lambda, I_\Lambda)$. Let $N = \langle N_s : s \in M(H) \rangle \trianglelefteq A$, where

$$N_s = \{(-i+j, gG') - (-i'+j', g'G') : M_s(i,j) = g\ M_s(i',j') = g'\}$$

- We are able to construct $\overline{f} : F_X \to A/N$ which gives the following result.

Theorem

The proabelian closure of H is $\ker \overline{f}$. Moreover, $\mathrm{Cl}(H)$ is finitely generated if, and only if, A/N is finite.