# Vertex-minimal graphs with non-abelian 2-group symmetry

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# Graphs with fixed symmetry

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A graph whose full automorphism group is isomorphic to $G$ is called a $G$-graph, and we let $\alpha(G)$ denote the minimal number of vertices among all $G$-graphs.

We compute $\alpha(G)$ when $G$ is isomorphic to either a quasi-dihedral group or a quasi-abelian group.

# The Automorphism Groups

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We need to construct a graph with $2^m$ vertices whose full group of automorphisms is isomorphic to $QD_{2^m}$.

$QD_{2^m}$ admits a Graphical Regular Representation (GRR), which is a Cayley Graph with no extra automorphisms, and we construct such a graph.

# Graph

Let $G$ be a group and suppose $S \subseteq G \backslash \{1\}$ is closed under inverses; the **Cayley graph** of $G$ with connection set $S$, denoted $Cay(G, S)$, is the graph with $V(Cay(G, S)) = G$ and

$$E(Cay(G, S)) = \{[g, gs] : g \in G \text{ and } s \in S\}.$$

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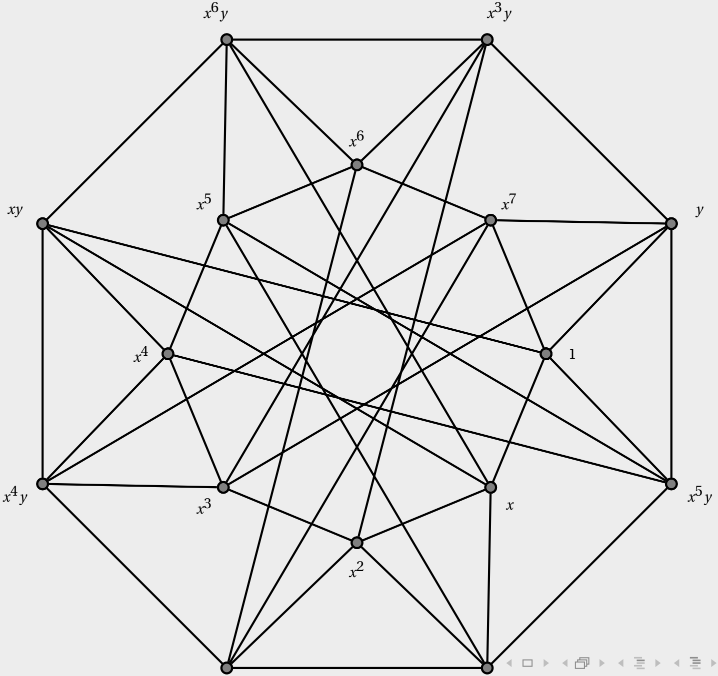
## Proposition

*Let $m \geq 4$. If $QD_{2^m} = \langle x, y \rangle$, then $Cay(QD_{2^m}, S)$ with connection set $S = \left\{x, x^{2^{m-1}-1}, y, xy, x^{2^{m-2}+1}y\right\}$ is a $QD_{2^m}$-graph.*


$x^6y$
$x^3y$
$x^6$
$xy$
$x^5$
$x^7$
$y$
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1
$x^4y$
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$x^2$

# Proof

Define the map $\pi_g : QD_{2^m} \to QD_{2^m}$ by $\pi_g(h) = gh$ where $h \in QD_{2^m}$. In this case, $\{\pi_g : g \in QD_{2^m}\}$ is a subgroup of $Aut(Cay(QD_{2^m}, S))$ that is isomorphic to $QD_{2^m}$.

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To prove the groups $\{\pi_g : g \in QD_{2^m}\}$ and $Aut(Cay(QD_{2^m}, S))$ are equal (i.e., that $Cay(QD_{2^m}, S)$ is a $QD_{2^m}$-graph) we apply the Orbit-Stabilizer Theorem.

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The Orbit-Stabilizer Theorem states that $|Aut(\Gamma)| = |Orbit(v)| \cdot |Stab(v)|$.

# Proof

Consider $x \in QD_{2^m}$. Since $Aut(Cay(QD_{2^m}, S))$ contains all left multiplications by elements in $QD_{2^m}$, it acts transitively on $V(Cay(QD_{2^m}, S)) = QD_{2^m}$.

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Now, we show that $Stab(x)$ is trivial and we are done.

Therefore, $\alpha(QD_{2^m}) \leq 2^m$.

# Lower Bound

## Proposition

*Let $\Gamma$ be a $QD_{2^m}$-graph with $Aut(\Gamma) = \langle \sigma, \tau \rangle$ as a permutation group where $\sigma, \tau$ generate $QD_{2^m}$. The cycle decomposition of $\sigma$ contains at least two $2^{m-1}$-cycles.*

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Suppose the cycle decomposition of $\sigma$ contains exactly one $2^{m-1}$-cycle.

WLOG $\sigma = (1,2,3,\ldots,2^{m-1})\rho$ where all cycles in $\rho$ have lengths less than $2^{m-1}$.

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$$2^m = |Aut(\Gamma)| = |Orbit(k)| \cdot |Stab(k)| = 2^{m-1} \cdot |Stab(k)|,$$

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It follows that $\alpha(QD_{2^m}) \geq 2^m$.

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This gives a contradiction.

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It follows that $\alpha(QD_{2^m}) \geq 2^m$.

Therefore, $\alpha(QD_{2^m}) = 2^m$ and we have proven the theorem.

# The Quasi-Abelian Group

## Theorem

*Let $m \geq 4$ be an integer. The quasi-abelian group $QA_{2^m}$ of order $2^m$ satisfies*

$$\alpha(QA_{2^m}) = \begin{cases} 18 & \text{if } m = 4 \\ 2^{m-1} + 6 & \text{if } m \geq 5. \end{cases}$$

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