π-submaximal subgroups of finite non-abelian simple groups

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Outline:

- Introduction to π-submaximal subgroups
- Introducting $\mathscr{S}$
- Four main theorems
- Classifying π-submaximal subgroups of elements of $\mathscr{S}$

Notation:

- For an integer $n > 1$, let $\pi(n)$ denote the set of prime divisors of n.
- For finite groups S, set $\pi(S) = \pi(|S|)$.
- Let π denote a set of primes. We use π' to denote the set of all primes not in π.
- We say H is a π-group if $\pi(H) \subseteq \pi$. We say that a subgroup of S is π-maximal if it is maximal as an element of the set $\mathcal{S} = \{H : H \text{ is a } \pi\text{-subgroup of } S\}$ partially ordered by inclusion.
- We use $m_\pi(S)$ to denote the set of all π-maximal subgroups of S.
- We say that H is a Hall π-subgroup of S if $|S : H|$ is a π'-number.
- We use $\mathrm{Hall}_\pi(S)$ to denote the set of all Hall π-subgroups of S.

- A subgroup H of S is *subnormal* (denoted $H \lhd\lhd S$) if there exists a collection of subgroups $\{H_i\}$ such that $H \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n \lhd S$.

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- In 1979, Wielandt introduced the concept of a π-submaximal subgroup.

Definition

Let π be a set of primes. A subgroup H of a finite group X is called a *π-submaximal subgroup* if there is a monomorphism $\varphi : X \to Y$ into a finite group Y such that $\varphi(X)$ is subnormal in Y and $\varphi(H) = K \cap \varphi(X)$ for some π-maximal subgroup K of Y. We say that a subgroup is proper π-submaximal if it is π-submaximal but not π-maximal.

- Given a group S, we will use $sm_\pi(S)$ to denote the set of π-submaximal subgroups of S. It is worth mentioning that $m_\pi(S) \subseteq sm_\pi(S)$ (just consider the identity map $S \to S$).

- If $H \in \mathrm{Hall}_\pi(S)$ and $N \lhd\lhd S$, then $H \cap N \in \mathrm{Hall}_\pi(N)$.
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- Issue with Hall π-subgroups: they may not exist.
- On the other hand, π-maximal subgroups will always exist. In the case of A_5, there are two isomorphism types of $\{3, 5\}$-maximal subgroups, which are the Sylow subgroups $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$.
- However, if $H \in m_\pi(S)$ and $N \lhd\lhd S$, it is not necessary that $H \cap N \in m_\pi(N)$.

- Consider the group S_7 which has a (sub)normal subgroup isomorphic to A_7. Let $\pi = \{2, 3, 7\}$.
- S_7 contains a π-maximal subgroup K of order 42.
- However, $K \cap A_7$ is a subgroup of A_7 of order 21 which is properly contained in a subgroup of A_7 isomorphic to $\mathrm{PSL}(3, 2)$.

Remark (Wielandt, 1979)

If H is a π-submaximal subgroup of S and $N \lhd\lhd S$, then $H \cap N$ is a π-submaximal subgroup of N.

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Lemma (Guo and Revin, 2017)

Let π be a set of primes. Then, for a subgroup H of a non-abelian simple group S, the following conditions are equivalent:

(1) H is a π-submaximal subgroup of S.

(2) There exists a group G such that S is the socle of G, G/S is a π-group, and $H = S \cap K$ for some K which is π-maximal in G.

(3) $H = S \cap K$ for some K which is π-maximal of $\mathrm{Aut}(S)$ where S is identified with $\mathrm{Inn}(S)$.

Definition

Let $\mathscr{S}$ denote the collection of simple groups $\mathscr{S} = \{\mathrm{PSL}(2, 2^e), \mathrm{PSL}(2, 3^e), \mathrm{PSL}(2, p), \mathrm{Sz}(2^e) :$ e is an odd prime and p is a prime greater than $5\}$.

Definition

We define $\mathscr{M} = \{M : M$ is isomorphic to a maximal subgroup of S for some $S \in \mathscr{S}\}$, and we define $\mathscr{D} \subseteq \mathscr{S} \times \mathscr{M}$ by $\mathscr{D} = \{(S, M) \in \mathscr{S} \times \mathscr{M}:$ M is isomorphic to a maximal subgroup of $S\}$.

Definition

Let $(S, M) \in \mathscr{D}$ and set $f(S) = (p, q, e)$. We eventually wish to define a map $\mathscr{G} : \mathscr{D} \to \mathcal{P}(\mathbb{P})$ by spanning the isomorphism types of elements of $\mathscr{M}$.

Theorem (Theorem 1)

Let $S \in \mathscr{S}$ and let M be a maximal subgroup of S. Suppose $\pi \in \mathscr{G}(S, M)$.

- $M \in \mathscr{C}_\pi$.
- *If $H \in Hall_\pi(M)$, then set $\pi_0 = \pi(H)$. Then $\pi_0 \in \mathscr{G}(S, M)$ and $H \in sm_\pi(S) \cap sm_{\pi_0}(S)$.*

Theorem (Theorem 2)

Let $S \in \mathscr{S}$ and let π be a set of primes such that $\pi(S) \nsubseteq \pi$ and $\pi \cap \pi(S) \neq \varnothing$. If $H \in sm_\pi(S)$, then there exists a maximal subgroup M of S such that $\pi \in \mathscr{G}(S, M)$ and $H \in Hall_\pi(M)$ except in the following counterexamples:

- *$S \cong PSL(2,7)$ and $H \cong D_6$, H is not a Hall π-subgroup of any maximal subgroup of S;*
- *$S \cong PSL(2,7)$ and $H \cong D_8$, H is not a Hall π-subgroup of any maximal subgroup of S;*
- *$S \cong PSL(2,11)$ and $H \cong D_{10}$, H is not a Hall π-subgroup of any maximal subgroup of S;*
- *$S \cong PSL(2,q)$ where q is prime and either $q \equiv \pm 11 (mod\ 40)$ or $q \equiv \pm 19 (mod\ 40)$ and $H \cong A_4$, H is a Hall $\{2,3\}$-subgroup of A_5, but H is not a Hall π-subgroup of A_5 if $5 \in \pi$.*

Theorem (Theorem 3)

Let π be a set of primes and let $S \in \mathscr{S}$. Let M be a maximal subgroup of S, and let $H \in m_\pi(M)$. If $\pi \in \mathscr{G}(S, M)$, then H is π-submaximal. If $\pi \notin \mathscr{G}(S, M)$ is such that $\pi \cap \pi(S) \neq \varnothing$, then H is not π-submaximal except possibly in the following instances:

- $H \cong C_3$ and $M \cong A_4$, $M \cong S_4$, or $M \cong A_5$;
- $H \cong C_5$ and $M \cong A_5$;
- $H \cong V_4$ and $M \cong A_4$ or $M \cong A_5$;
- $H \cong D_6$ and $M \cong A_5$;
- $H \cong D_{10}$ and $M \cong A_5$;
- $H \cong D_8$ and $M \cong S_4$.

Theorem (Theorem 4)

Let $S \in \mathscr{S}$ be a simple group and let $H \neq 1$ be a proper subgroup of S. For any maximal subgroup M of S which contains H, set $\pi_M(H) = \{\pi : H \text{ is a Hall } \pi\text{-subgroup of } M\}$. Set

$$\Pi = \bigcup_{\substack{M \lessdot S \\ H \leq M}} \left(\pi_M(H) \cap \mathscr{G}(S, M)\right).$$

Then $H \in sm_\pi(S)$ if and only if $\pi \in \Pi$ unless we are in one of the following cases:

- $H \cong A_4$, $p \equiv \pm 11 (mod\ 40)$ *or* $p \equiv \pm 19 (mod\ 40)$, *and* $\{2, 3, 5\} \subseteq \pi \notin \Pi$.
- $H \cong D_6$, $p = 7$, $\{2, 3\} \subseteq \pi$, *and* $\Pi = \varnothing$;
- $H \cong D_8$, $p = 7$, $\{2\} \subseteq \pi$, *and* $\Pi = \varnothing$;
- $H \cong D_{10}$, $p = 11$, $\{2, 5\} \subseteq \pi$, *and* $\Pi = \varnothing$.

Definition

We define the following subsets of the power set of the prime integers $\mathcal{P}(\mathbb{P})$. Let p, k be primes, q be a prime power. Let $S \in \mathscr{S}$.

- $\mathcal{S}_p = \{\pi \in \mathcal{P}(\mathbb{P}) : p \in \pi\}$. We set $\mathcal{S}_{p'} = \mathcal{P}(\mathbb{P}) \setminus \mathcal{S}_p$.
- $\mathcal{C}_{(q,+)}$ by $\mathcal{C}_{(q,+)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q+1) \neq \varnothing\}$.
- $\mathcal{C}_{(q,-)}$ by $\mathcal{C}_{(q,-)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q-1) \neq \varnothing\}$.
- $\mathcal{D}_{(q,+)}$ by $\mathcal{D}_{(q,+)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q+1) \neq \{2\}\}$.
- $\mathcal{D}_{(q,-)}$ by $\mathcal{D}_{(q,-)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q-1) \neq \{2\}\}$.
- $\mathcal{R}_{(q,+)}$ by $\mathcal{R}_{(q,+)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q+\sqrt{2q}+1) \neq \varnothing\}$.
- $\mathcal{R}_{(q,-)}$ by $\mathcal{R}_{(q,-)} = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \cap \pi(q-\sqrt{2q}+1) \neq \varnothing\}$.
- $\mathcal{O}_S$ by $\mathcal{O}_S = \{\pi \in \mathcal{P}(\mathbb{P}) : \pi \not\supseteq \pi(S)\}$.

Definition

If $M \cong E_q \rtimes C_{\left(\frac{1}{k_q}(q-1)\right)}$, we set

$$\mathscr{G}(S, M) = \left(\mathcal{S}_p \cup \left(\mathcal{C}_{(q,-)} \cap \mathcal{S}_{2'}\right)\right) \cap \mathcal{O}_S.$$

Definition

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Definition

If $M \cong A_5$, we set

$$\mathscr{G}(S,M) = \begin{cases} (\mathcal{S}_2 \cap \mathcal{S}_3) \cap \mathcal{O}_S & \text{if } p \equiv \pm 11, \pm 19 (\text{mod } 40) \\ (\mathcal{S}_2 \cap \mathcal{S}_3 \cap \mathcal{S}_5) \cap \mathcal{O}_S & \text{if } p \not\equiv \pm 11, \pm 19 (\text{mod } 40) \end{cases}$$

Definition

If $M \cong D_{\left(\frac{2(q+1)}{kq}\right)}$, we set

$$\mathscr{G}(S,M) = \begin{cases} \mathcal{C}_{(q,+)} \cap \mathcal{O}_S & \text{if } p = 2 \\ \left(\left(\mathcal{C}_{(q,+)} \cap \mathcal{D}_{(q,+)}\right) \cup (\mathcal{S}_2 \cap \mathcal{S}_{3'})\right) \cap \mathcal{O}_S & \text{if } p = 3 \\ \left(\left(\mathcal{C}_{(q,+)} \cap \mathcal{S}_{p'} \cap \mathcal{S}_{2'}\right) \cup \left(\mathcal{S}_2 \cap \left(\mathcal{D}_{(q,+)} \cup \left(\mathcal{S}_{3'}^{(p,8,+)} \cap \mathcal{S}_{5'}^{(p,10,+)} \cap \varnothing^{(p,4,+)}\right)\right)\right)\right) \cap \mathcal{O}_S & \text{if } p > 3 \end{cases}$$

Definition

If $M \cong D_{\left(\frac{2(q-1)}{kq}\right)}$, we set

$$\mathscr{G}(S,M) = \begin{cases} \mathcal{C}_{(q,-)} \cap \mathcal{O}_S & \text{if } p = 2 \\ \left(\mathcal{C}_{(q,-)} \cap \mathcal{D}_{(q,-)}\right) \cap \mathcal{O}_S & \text{if } p = 3 \\ \left(\left(\mathcal{C}_{(q,-)} \cap \mathcal{S}_{p'} \cap \mathcal{S}_{2'}\right) \cup \left(\mathcal{S}_2 \cap \left(\mathcal{D}_{(q,-)} \cup \left(\mathcal{S}_{3'}^{(p,8,-)} \cap \mathcal{S}_{5'}^{(p,10,-)} \cap \varnothing^{(p,4,-)}\right)\right)\right)\right) \cap \mathcal{O}_S & \text{if } p > 3 \end{cases}$$

Definition

If $M \cong (E_q \times E_q) \rtimes C_{(q-1)}$, we set

$$\mathscr{G}(S, M) = \left(\mathcal{S}_2 \cup \mathcal{C}_{(q,-)}\right) \cap \mathcal{O}_S.$$

Definition

If $M \cong C_{(q-\sqrt{2q}+1)} \rtimes C_4$, we set

$$\mathscr{G}(S, M) = \mathcal{R}_{(q,-)} \cap \mathcal{O}_S.$$

Definition

If $M \cong C_{(q+\sqrt{2q}+1)} \rtimes C_4$, we set

$$\mathscr{G}(S, M) = \mathcal{R}_{(q,+)} \cap \mathcal{O}_S.$$

Theorem (π-submaximal subgroups of PSL$(2, 2^e)$)

Let $q = 2^e$ for some odd prime e and let $S \cong PSL(2, 2^e)$. Let π be a set of primes such that $\pi(S) \not\subseteq \pi$ and $\pi(S) \cap \pi \neq \varnothing$. Let H be a π-submaximal subgroup of S. Then H is π-maximal and isomorphic to the following subgroups under the following conditions on π. In addition, the pronormality and number of conjugacy classes (NCC) for each subgroup is also given.

H	Conditions	Pronormal	NCC
$C_{(q-1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{C}_{(q,-)}$	✓	1
$C_{(q+1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{C}_{(q,+)}$	✓	1
$E_q \rtimes C_{(q-1)_\pi}$	$\pi \in \mathcal{S}_2$	✓	1
$D_{2(q-1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \mathcal{C}_{(q,-)}$	✓	1
$D_{2(q+1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \mathcal{C}_{(q,+)}$	✓	1

Theorem (π-submaximal subgroups of $\mathrm{PSL}(2, 3^e)$)

Let $q = 3^e$ for some odd prime r and let $S \cong PSL(2, 3^e)$. Let π be a set of primes such that $\pi(S) \not\subseteq \pi$ and $\pi \cap \pi(S) \neq \varnothing$. Let H be a π-submaximal subgroup of S. Then H is π-maximal and isomorphic to one of the following subgroups under the following conditions on π. In addition, the pronormality and number of conjugacy classes for each subgroup is also given.

H	Conditions	Pronormal	NCC
$C_{\frac{1}{2}(q-1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{S}_{3'} \cap \mathcal{C}_{(q,-)}$	✓	1
$C_{\frac{1}{2}(q+1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{C}_{(q,+)}$	✓	1
$E_q \rtimes C_{\frac{1}{2}(q-1)_\pi}$	$\pi \in \mathcal{S}_3$	✓	1
$D_{(q-1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \mathcal{D}_{(q,-)}$	✓	1
$D_{(q+1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \left(\mathcal{D}_{(q,+)} \cup \mathcal{S}_{3'}\right)$	✓	1
A_4	$\pi \in \mathcal{S}_2 \cap \mathcal{S}_3$	✓	1

Theorem (π-submaximal subgroups of $PSL(2, p)$ where $p > 5$ is prime)

Let $p > 5$ be a prime integer and let $S \cong PSL(2, p)$. Let π be a set of primes such that $\pi(S) \not\subseteq \pi$ and $\pi \cap \pi(S) \neq \varnothing$. Let H be a π-submaximal subgroup of S. Then H is isomorphic to the following subgroups under the following conditions on π. In addition, the number of conjugacy classes (NCC) of each subgroup type and pronormality is also given.

H	*Conditions on π and p*	*H not π-maximal when*	*Pronormal*	*NCC*
$C_{\frac{1}{2}(p-1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{S}_{p'} \cap \mathcal{C}_{(p,-)}$		✓	*1*
$C_{\frac{1}{2}(p+1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{C}_{(p,+)}$		✓	*1*
$C_p \rtimes C_{\frac{1}{2}(p-1)_\pi}$	$\pi \in \mathcal{S}_p$		✓	*1*
$D_{(p-1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \left(\mathcal{D}_{(p,-)} \cup \left(\mathcal{S}_{3'}^{(p,8,-)} \cap \mathcal{S}_{5'}^{(p,10,-)} \cap \varnothing^{(p,4,-)}\right)\right)$	$\pi \in \mathcal{D}_{(p,-)} \cap \mathcal{S}_3$ *and* $p \equiv 41(\textit{mod } 48)^{*}$ *OR* $\pi \in \mathcal{X}_{(p,3,-)}$ *and* $p \equiv 7, 31(\textit{mod } 72)^{**}$ *OR* $\pi \in \mathcal{X}_{(p,3,-)} \cap \mathcal{S}_5$ *and* $p \equiv 79, 139(\textit{mod } 180)^{***}$ *OR* $\pi \in \mathcal{X}_{(p,5,-)} \cap \mathcal{S}_3$ *and* $p \equiv 11, 71, 131, 191(\textit{mod } 300)^{****}$	✓	*1*
$D_{(p+1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \left(\mathcal{D}_{(p,+)} \cup \left(\mathcal{S}_{3'}^{(p,8,+)} \cap \mathcal{S}_{5'}^{(p,10,+)} \cap \varnothing^{(p,4,+)}\right)\right)$	$\pi \in \mathcal{D}_{(p,+)} \cap \mathcal{S}_3$ *and* $p \equiv 7(\textit{mod } 48)^{*}$ *OR* $\pi \in \mathcal{X}_{(p,3,+)}$ *and* $p \equiv 41, 65(\textit{mod } 72)^{**}$ *OR* $\pi \in \mathcal{X}_{(p,3,+)} \cap \mathcal{S}_5$ *and* $p \equiv 41, 101(\textit{mod } 180)^{***}$ *OR* $\pi \in \mathcal{X}_(p, 5, +) \cap \mathcal{S}_3$ *and* $p \equiv 109, 169, 229, 289(\textit{mod } 300)^{****}$	✓	*1*
A_4	$\pi \in \mathcal{S}_2 \cap \mathcal{S}_3$ *and* $p \equiv \pm 3, 11, 13, 19(\textit{mod } 40)$	$\pi \in \mathcal{S}_5$ *and* $p \equiv \pm 11, 19(\textit{mod } 40)^{\bowtie}$	✓	*1*
S_4	$\pi \in \mathcal{S}_2 \cap \mathcal{S}_3$ *and* $p \equiv \pm 1(\textit{mod } 8)$		✓	*2*
A_5	$\pi \in \mathcal{S}_2 \cap \mathcal{S}_3 \cap \mathcal{S}_5$ *and* $p \equiv \pm 1(\textit{mod } 10)$		✓	*2*

Theorem (π-submaximal subgroups of $Sz(2^e)$)

Let $S \cong Sz(2^e)$ for some odd prime e. Set $q = 2^e$ and let π be a set of primes such that $\pi(S) \nsubseteq \pi$ and $\pi \cap \pi(S) \neq \varnothing$. Let H be a π-submaximal subgroup of S. Then H is π-maximal and isomorphic to the following subgroups under the following conditions on π. In addition, the pronormality and number of conjugacy classes (NCC) for each subgroup is also given.

M	*Conditions*	*Pronormal*	*NCC*
$C_{(q-1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{C}_{(q,-)}$	✓	*1*
$C_{(q-\sqrt{2q}+1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{R}_{(q,-)}$	✓	*1*
$C_{(q+\sqrt{2q}+1)_\pi}$	$\pi \in \mathcal{S}_{2'} \cap \mathcal{R}_{(q,+)}$	✓	*1*
$(E_q \times E_q) \rtimes C_{(q-1)_\pi}$	$\pi \in \mathcal{S}_2$	✓	*1*
$D_{2(q-1)_\pi}$	$\pi \in \mathcal{S}_2 \cap \mathcal{C}_{(q,-)}$	✓	*1*
$C_{(q-\sqrt{2q}+1)_\pi} \rtimes C_4$	$\pi \in \mathcal{S}_2 \cap \mathcal{R}_{(q,-)}$	✓	*1*
$C_{(q+\sqrt{2q}+1)_\pi} \rtimes C_4$	$\pi \in \mathcal{S}_2 \cap \mathcal{R}_{(q,+)}$	✓	*1*