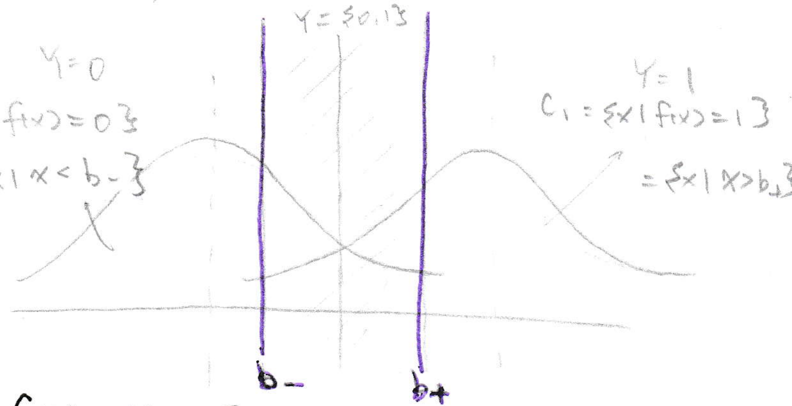
# Classification with Confidence




- A New Framework of classification

  $x \in \mathcal{X}$, $Y \in \{0,1\}$, $(X, Y) \sim P(X, Y)$

  want to find a classification rule $f(x): \mathcal{X} \to \{\{0\}, \{1\}, \{0,1\}\}$

  - define
    $$\left.\begin{aligned} C_0 &= \{x \mid f(x) = 0\} \\ C_1 &= \{x \mid f(x) = 1\} \end{aligned}\right\} \quad C_0 \cup C_1 = \mathcal{X}$$

    $C_{01} = C_0 \cap C_1$ is where the data could be "classified" to $\{0,1\}$, which called "Ambiguity"

  - new question becomes a trade-off b/w classification accuracy and the portion of ambiguity.

    → want to minimize the ambiguity portion w/ a control on accuracy, i.e.

(*) ↑ target

given
$$P_0(C_0) = P(X \in C_0 \mid Y = 0) \geq 1 - \alpha_0 \quad \leftarrow$$
$$P_1(C_1) = P(X \in C_1 \mid Y = 1) \geq 1 - \alpha_1 \quad \leftarrow$$
accuracies

and $C_0 \cup C_1 = \mathcal{X}$

want to minimize

$$P(C_0 \cap C_1) = \pi_1 P_1(C_0 \cap C_1) + \pi_0 P_0(C_0 \cap C_1)$$

$\pi_j = P(Y = j) \quad j = 0, 1$

- Thm 1 gives a solution to ~~probl~~ the target problem (*)

  Thm 1 says, fix $0 < \alpha_0, \alpha_1 < 1$ a solution to (*) is given by

  $$C_0 = \{x : \eta(x) \leq t_0(\alpha_0)\}, \quad C_1 = \{x : \eta(x) \geq t_1(\alpha_1)\} \cup C_0^c$$

  where $t_0$ and $t_1$ are chosen s.t.

  $$P_0(\eta(x) \leq t_0(\alpha_0)) = 1 - \alpha_0, \quad P_1(\eta(x) \geq t_1(\alpha_1)) = 1 - \alpha_1$$

- quick proof of Thm 1.

**Proof:**

WoLG. assume $t_0 \geq t_1$

o.w. $C_0 \cap C_1 = \emptyset$

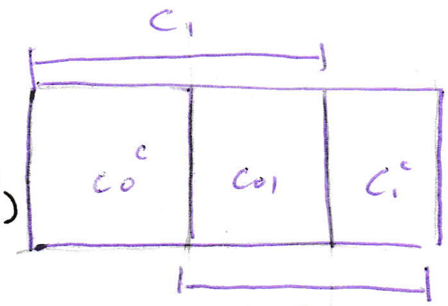
Suppose we have a new classifier $f^*(x)$ w/

$S_0 = \{x \mid f^*(x) = 0\}$, $P_0(S_0) \geq 1-\alpha_0$

$S_1 = \{x \mid f^*(x) = 1\}$, $P_1(S_1) \geq 1-\alpha_1$

$S_{01} = S_0 \cap S_1$

<u>want to show</u> $P_0(C_{01}) \leq P_0(S_{01})$




$\because P_0(C_0) \leq P_0(S_0)$

$\Rightarrow P_0(C_1^c \cup C_{01}) \leq P_0(S_0 \cap C_0) + P_0(S_0 \cap C_0^c)$

$\Rightarrow P_0[(C_1^c \cup C_{01}) \cap S_0] + P[(C_1^c \cup C_{01}) \cap S_0^c] \leq P[(C_{01} \cup C_1^c) \cap S_0] + P_0(S_0 \cap C_0^c)$

$\Rightarrow P_0(\cancel{C_1^c \cap S_0}) + P_0(\cancel{C_{01} \cap S_0}) + P_0(C_1^c \cap S_0^c) + P_0(C_{01} \cap S_0^c)$

$\leq P_0(\cancel{C_{01} \cap S_0}) + P_0(\cancel{C_1^c \cap S_0}) + P_0(\underbrace{S_0}_{= S_{01} \cup S_1^c} \cap C_0^c)$

$\Rightarrow P_0(C_1^c \cap S_0^c) + P_0(C_{01} \cap S_0^c) \leq P_0(S_{01} \cap C_0^c) + P_0(S_1^c \cap C_0^c)$

$\Rightarrow P_0(C_{01} \cap S_0^c) - P_0(C_0^c \cap S_{01}) \leq P_0(S_1^c \cap C_0^c) - P_0(S_0^c \cap C_1^c)$ (**)

$\therefore P_0(C_{01}) - P_0(S_{01})$

$= P_0(C_{01} \cap S_1^c) + \overbrace{P_0(C_{01} \cap S_0^c) - P_0(C_0^c \cap S_{01})}^{\text{l.h.s. of (**)}} - P_0(C_1^c \cap S_{01})$

$\leq P_0(C_{01} \cap S_1^c) + P_0(S_1^c \cap C_0^c) - P_0(S_0^c \cap C_1^c) - P_0(C_1^c \cap S_{01})$

$= P_0(C_1 \cap S_1^c) - P_0(S_1 \cap C_1^c)$

$\therefore P_0(C_{01}) - P_0(S_{01}) \le P_0(\underbrace{C_1 \cap S_1^c}_{\eta(x) \ge t_1(\alpha_1) \text{ in this set}}) - P_0(\underbrace{S_1 \cap C_1^c}_{\eta(x) \le t_0(\alpha_0) \text{ in this set}})$

b/c $\forall x \in C_1^c \Rightarrow x \in C_0$

If we consider a discrete $X$

$$\therefore \text{for } \forall x \in C_1 \cap S_1^c, \quad \frac{P(Y=1 \mid X=x)}{P(Y=0 \mid X=x)} = \frac{\eta}{1-\eta} \ge \frac{t_1(\alpha_1)}{1-t_1(\alpha_1)}$$

$$\frac{P(Y=1, X=x)/P(X=x)}{P(Y=0, X=x)/P(X=x)} = \frac{P(Y=1, X=x)/P(Y=1)}{P(Y=0, X=x)/P(Y=0)} \cdot \frac{\overbrace{P(Y=1)}^{\pi_1}}{\underbrace{P(Y=0)}_{\pi_0}}$$

$$= \frac{P_1(X)}{P_0(X)} \cdot \frac{\pi_1}{\pi_0} \ge \frac{t_1}{1-t_1}$$

$$\Rightarrow P_0(X) \le \frac{1-t_1}{t_1} \cdot \frac{\pi_1}{\pi_0} P_1(X)$$

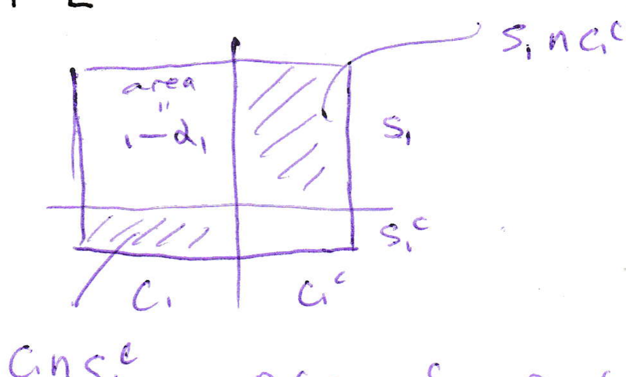
$$\therefore P_0(C_1 \cap S_1^c) \le \frac{1-t_1}{t_1} \cdot \frac{\pi_1}{\pi_0} P_1(C_1 \cap S_1^c)$$

Similarly, $P_0(S_1 \cap C_1^c) \le \frac{1-t_1}{t_1} \cdot \frac{\pi_1}{\pi_0} P_1(S_1 \cap C_1^c)$

$$\therefore P_0(C_{01}) - P_0(S_{01}) \le \frac{\pi_1}{\pi_0} \frac{1-t_1}{t_1} \left[ P_1(C_1 \cap S_1^c) - P_1(S_1 \cap C_1^c) \right]$$

$\therefore P_0(C_{01}) - P_0(S_{01}) \le 0$

b/c
$P_1(C_1 \cap S_1^c) \le P_1(S_1 \cap C_1^c)$




$P_1(C_1 \cap S_1^c) \le P_1(C_1^c \cap S_1)$

• If $\tilde{\eta}$ is estimated by $\hat{\eta}$,

then classifications can be estimated as

$$\hat{C}_0 = \{x : \hat{\eta}(x) \leq \hat{t}_0(\alpha_0)\} \qquad \hat{C}_1 = \{x : \hat{\eta}(x) \geq \hat{t}_1(\alpha_1)\} \cup \hat{C}_0^c$$

estimate $t_0$ and $t_1$ depends on how to model the data

If order all $\hat{\eta}(x_i)$'s, then $\alpha_0$, $\alpha_1$ can get controled by choosing sample quantiles.

- Want the estimate $\hat{\eta}$ to be $\boxed{\text{good}}$
- If $G_i$ is c.d.f of $\eta(x)$ under $P_i$

(***) want $b_1|\varepsilon|^\delta \leq |G_i(t_i+\varepsilon) - G_i(t_i)| \leq b_2|\varepsilon|^\delta$

$G_i$ should be smooth at true $t_i$'s

• $(\delta_n, \rho_n)$ accuracy

An estimator $\hat{\eta}$ is $(\delta_n, \rho_n)$ accurate if $P(\|\hat{\eta}-\eta\| \geq \delta_n) \leq \rho_n$

• Thm 2

If $\hat{\eta}$ is $(\delta_n, \rho_n)$-accurate and condition (***) holds, then for each $r > 0$, $\exists$ a positive constant $c$ s.t.

$$P_j(\hat{C}_j \Delta C_j) \leq c\left\{\delta_n^{\nu} + \left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right\}, \quad j = 0, 1$$

w/ probability at least $1 - \rho_n - n^{r}$

i.e. $L_1$-panelized logistic regression gives $\delta_n$-$\rho_n$ accurate estimator.