Problem 7. Consider an $m \times n$ rectangle divided into mn unit squares. Let T be the set of all vertices of the unit squares. At each point of T we draw a short arrow (say of length $1/2$) pointing up, down, left, or right with the condition that no arrow sticks outside the rectangle. Prove that regardless of how the arrows are chosen, there always must exist two vertices of the same unit square at which the arrows point in opposite directions.

Solution. We assume that our rectangle is in the $x-y$ plane of $\mathbb{R}^3$ and has vertices $A = (0,0,0)$, $B = (m,0,0)$ $C = (m,n,0)$, $D = (0,n,0)$. We also assume that the arrows at each point of T are unit vectors, i.e. at each point we have one of the vectors $\pm\mathbf{i}$, $\pm\mathbf{j}$, where $\mathbf{i} = [1,0,0]$ and $\mathbf{j} = [0,1,0]$. Let $\mathbf{k} = [0,0,1]$.

We will consider paths along the edges of the unit squares. We define **the cost** of moving from vertex X to vertex Y of an edge XY of a unit square to be the vector $\mathbf{u} \times \mathbf{w}$, where $\mathbf{u}$ is the unit vector assigned to X, $\mathbf{w}$ is the unit vector assigned to Y, and $\times$ denotes the cross product (see the third solution to Problem 1 from Fall 2025 for a discussion of cross product). Recall that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{u} \times \mathbf{w} = -\mathbf{w} \times \mathbf{u}$, and $(-\mathbf{u}) \times \mathbf{w} = -\mathbf{u} \times \mathbf{w} = \mathbf{u} \times -\mathbf{w}$. It follows that the cost of moving from X to Y is $\pm\mathbf{k}$, and the cost of moving from Y to X is the negative of the cost of moving from X to Y.

For any path along the edges of the unit squares which starts and ends in T we define **the cost** of the path to be the sum of the costs of moving through each of the edges in the path. The key to our solution is the following observation:

The cost of moving along the boundary of the rectangle $ABCD$ counterclockwise is equal to the sum of costs of moving counterclockwise along the boundary of each unit square.

Indeed, note that each edge XY on the boundary of the rectangle $ABCD$ belongs to unique unit square and each edge inside the rectangle $ABCD$ belongs to exactly 2 unit squares. Furthermore if moving counterclockwise around one unit square containing XY we travel from X to Y, then the counterclockwise travel around the second unit square containing XY moves from Y to X. Thus, when we add the costs of moving counterclockwise along the boundary of each unit square, the costs incurred on edges inside the rectangle cancel out and we are left with the cost of moving along the boundary of the rectangle $ABCD$ counterclockwise.

Suppose now that there do not exist two vertices of the same unit square at which the arrows point in opposite directions. This means that in any unit square $XYZW$ the unit vectors at its vertices satisfy one of the following properties:

1. all unit vectors are equal to same unit vector $\mathbf{u} \in \{\pm\mathbf{i}, \pm\mathbf{j}\}$.
2. three of the unit vectors are equal to $\mathbf{u} \in \{\pm\mathbf{i}, \pm\mathbf{j}\}$ and the forth one is $\mathbf{w} \in \{\pm\mathbf{i}, \pm\mathbf{j}\}$, where $\mathbf{u}$ and $\mathbf{w}$ are perpendicular.
3. two of the units vectors are equal to $\mathbf{u} \in \{\pm\mathbf{i}, \pm\mathbf{j}\}$ and the other two are equal to $\mathbf{w} \in \{\pm\mathbf{i}, \pm\mathbf{j}\}$, where $\mathbf{u}$ and $\mathbf{w}$ are perpendicular.

We claim that in each case, the cost P of of moving counterclockwise along the boundary of $XYZW$ is 0. Indeed, in case 1. the cost of each edge is $\mathbf{u} \times \mathbf{u} = \mathbf{0}$, so clearly $P = 0$. In case 2.,

$$P = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} = \mathbf{0}.$$

In case 3. we either have two consecutive vertices with $\mathbf{u}$ and two consecutive vertices with $\mathbf{w}$ so the cost is

$$P = \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{w} + \mathbf{w} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} = \mathbf{0},$$

or both $\mathbf{u}$ and $\mathbf{w}$ are assigned to vertices on a diagonal of the unit square and the cost is

$$\mathbf{u} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} + \mathbf{u} \times \mathbf{w} + \mathbf{w} \times \mathbf{u} = \mathbf{0}.$$

This completes the justification of our claim. From our key observation, we conclude that the cost of moving along the boundary of the rectangle $ABCD$ counterclockwise is 0.

We will derive a contradiction by showing that the cost of moving along the boundary of $ABCD$ can not be 0. To this end, let us consider the path along any edge of the rectangle $ABCD$ when moving counterclockwise. Without any loss of generality we focus on the path AB. Since no arrow sticks outside the rectangle, at each vertex in T which is on the side AB the unit vector assigned to this vertex is $\pm\mathbf{i}$ or $\mathbf{j}$. Also, A has assigned either $\mathbf{i}$ or $\mathbf{j}$ and B has assigned either $-\mathbf{i}$ or $\mathbf{j}$. Suppose that no vertex on AB has assigned $\mathbf{j}$. Then when moving from A to B we start with vertex having assigned $\mathbf{i}$ and end with vertex having assigned $-\mathbf{i}$. It follows that at some point two consecutive vertices have assigned opposite unit vectors, contrary to our assumption. Thus this case is not possible. Suppose that when moving from A to B the vertices which have assigned unit vector $\mathbf{j}$ are $V_1, \ldots, V_k$. Note that each vertex between V_s and V_{s+1} has assigned either $\mathbf{i}$ or $-\mathbf{i}$. Since no two consecutive vertices have opposite vectors assigned to them, we conclude that all vertices between V_s and V_{s+1} have assigned the same unit vector $\mathbf{u} = \pm\mathbf{i}$. This means that the cost of traveling from V_s to V_{s+1} is

$$\mathbf{j} \times \mathbf{u} + \mathbf{u} \times \mathbf{u} + \ldots + \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{j} = \mathbf{0}$$

(this works even when there are no vertices between V_s and V_{s+1}). It follows that the cost of the path AB is equal to the cost of the path AV_1 plus the cost of the path V_kB. If $A \neq V_1$, then all vertices before V_1 have vector $\mathbf{i}$ assigned to them and the cost of the path AV_1 is $\mathbf{i} \times \mathbf{i} + \ldots + \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similarly, if $V_k \neq B$ then all vertices after V_k have vector $-\mathbf{i}$ assigned to them and the cost of the path V_kB is again $\mathbf{k}$. It follows that

$$\text{the cost of the path } AB = \begin{cases} 0 & \text{if vectors assigned to both } A,\, B \text{ are perpendicular to } AB \\ \mathbf{k} & \text{if exactly one of the vectors assigned to } A,\, B \text{ is perpendicular to } AB \\ 2\mathbf{k} & \text{if neither of the vectors assigned to } A,\, B \text{ is perpendicular to } AB. \end{cases}$$

The same holds for each side of the rectangle. It follows that the only way that the cost of going around the boundary of $ABCD$ counterclockwise is 0 is when the cost of each side of $ABCD$ is 0. However this is not possible, as the vector assigned to A can not be both perpendicular to AB and AD. Therefore the cost of going around $ABCD$ counterclockwise cannot be 0, a contradiction.

Exercise. Show that the cost of moving around $ABCD$ counterclockwise in the analysis above is actually equal to $4\mathbf{k}$.

Second solution (after Matt Wolak). The solution submitted by Matt Wolak is similar in spirit to our solution, but it my be easier to follow.

Suppose that no two vertices that share an edge have arrows that point in opposite directions. Then for any path from vertex to vertex along the edges of unit squares we can define an **index** by starting from zero, then adding

- 0, when you travel from one vertex to another whose arrow points in the same direction,
- 1/4, if when you travel from one vertex to another whose arrow is 90° counterclockwise from the old one,
- −1/4 if when you travel from one vertex to another whose arrow is 90° clockwise from the old one.

If a path starts and ends on the same vertex, the index must be an integer, since you must have turned around a whole number of times to get back to the starting orientation. Traversing a path backwards will negate the index, and concatenating two paths gives the sum of the indices. The index of the path that goes counterclockwise around the perimeter of the rectangle must be 1. This is because no arrow along the top can point up, no arrow along the left edge can point left, no arrow along the bottom can point down, and no point along the right side can point right. In order to avoid these forbidden directions, you must make a single rotation counterclockwise along that path. Finally, the index of the path that goes counterclockwise around the perimeter of the rectangle is equal to the sum of the indices of the paths that go counterclockwise around each unit square. In the sum, each internal edge is traversed in both directions, so its contributions cancel out. This implies that at least one of the unit squares has index equal to 1, which in turn implies that it has an arrow pointing in each of the four directions.

Exercise. Carefully justify all claims made in the above solution.

Third solution (after Levi Axelrod). We assume that our rectangle has vertices $A = (0, 0)$, $B = (m, 0)$ $C = (m, n)$, $D = (0, n)$ and that the vectors in each point of T are the unit vectors from $\{\pm\mathbf{i}, \pm\mathbf{j}\}$, where $\mathbf{i} = [\mathbf{1}, \mathbf{0}]$ and $\mathbf{j} = [\mathbf{0}, \mathbf{1}]$. Suppose that there do not exist two vertices of the same unit square at which the arrows point in opposite directions. In each unit square we draw one of the diagonals. Then the rectangle is triangulated into triangles. The vertices of each triangle either all have the same unit vector $\mathbf{u}$ assigned or two of them have the same unit vector $\mathbf{u}$ assigned and the third has assigned a vector $\mathbf{w}$ perpendicular to $\mathbf{u}$. Consider one of these triangles, say XYZ. Every point P in the triangle XYZ can be written in a unique way as $P = pX + sY + tZ$ for some non-negative numbers p, s, t such that $p + s + t = 1$. We assign to the point P the vector $\mathbf{u_P} = \mathbf{pu_X} + \mathbf{su_Y} + \mathbf{tu_Z}$, where $\mathbf{u_X}$ is the unit vector assigned to X, etc.. We claim that u_P is never 0. In fact, if $\mathbf{u_X} = \mathbf{u_Y} = \mathbf{u_Z} = \mathbf{u}$ then $\mathbf{u_P} = \mathbf{u}$. If $\mathbf{u_X} = \mathbf{u_Y} = \mathbf{u}$ and $\mathbf{u_Z} = \mathbf{w}$ then $\mathbf{u_P} = (\mathbf{p} + \mathbf{s})\mathbf{u} + \mathbf{tw} \neq \mathbf{0}$. Note that if a point belongs to two different triangles, then the vector assigned to it in each of the triangles is the same. It is now easy to see that we assigned to each point P of the rectangle a non-zero vector $V(P)$ which depends continuously on P. The fact that no arrow assigned to vertices in T sticks outside the rectangle implies that for $\epsilon > 0$ small enough the point $P + \epsilon V(P)$ belongs to the rectangle for all points P. Thus the assignment $P \mapsto P + \epsilon V(P)$ defines a continuous function from the rectangle to itself without any fixed points. This however contradicts the celebrated theorem called **Brouwer's fixed-point theorem**, which asserts that every continuous function from a rectangle to itself must have a fixed point.

Exercise. Carefully justify all the claims in the above solution (you do not need to justify Brouwer's theorem).

Remark. The solution submitted by Levi actually has a slightly different argument. Levi considers the function $P \mapsto V(P)/|V(P)|$ from the square to the circle. He argues that this map restricted to the boundary of the rectangle, which can be identified with the circle, is homotopic to the identity map on the circle. On the other hand, since it extends to the whole rectangle, it is homotopic to the constant map. A basis result in algebraic topology says that the identitty map on the circle is not homotopic to the constant map, so we get a contradiction. Filling all the details in this approach seems more difficult than in the one outlined above.

Exercise. This exercise derives Brouwer's fixed point theorem from Problem 7. Suppose that $F : [0, 1] \times [0, 1] \longrightarrow [0, 1] \times [0, 1]$ is a continuous function without fixed points (i.e. $F(P) \neq P$ for all P). We write $F(x, y) = (f(x, y), g(x, y))$. Let $U_+ = \{(x, y) : f(x, y) > x\}$, $U_- = \{(x, y) : f(x, y) < x\}$, $V_+ = \{(x, y) : g(x, y) > y\}$, $V_- = \{(x, y) : g(x, y) < y\}$.

a) Show that U_+, U_-, V_+, V_- are open sets and each point of the square $[0, 1] \times [0, 1]$ belongs to at least one and at most two of these sets.

b) Divide the square into n^2 squares of size $1/n \times 1/n$. Let (x, y) be one of the vertices of these little squares. If $f(x, y) \neq x$ then assign to (x, y) the vector $\epsilon\mathbf{i}/\mathbf{n}$, where ϵ is the sign of $f(x, y) - x$. If $f(x, y) = x$ then $g(x, y) \neq y$ and assign to (x, y) the vector $\epsilon\mathbf{j}/\mathbf{n}$, where ϵ is the sign of $g(x, y) - y$. Show that no arrow assigned in this way sticks outside the square.

c) Prove that if the n in b) is sufficiently large then each little square is contained in one of the sets U_+, U_-, V_+, V_-. Conclude that no two vertices of the same little square have arrows pointing in opposite directions. Derive a contradiction.