Problem 7. Prove that for every $n \geq 1$ the number

$$\frac{(1^2 + 2^2 + \ldots + n^2)!}{(1!)^2 \cdot (2!)^3 \cdot (3!)^4 \cdot \ldots \cdot (n!)^{n+1}}$$

is an integer.

Solution. Recall than for any integers $n \geq k \geq 0$ the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is an integer. This can be proved by induction using the formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Alternatively, one can use the combinatorial interpretation: $\binom{n}{k}$ is equal to the number of subsets of size k of a set of size n.

Our solution is based on the following simple observation:

$$\binom{nk}{k} = \frac{(nk)!}{k!(nk-k)!} = \frac{(nk-1)!nk}{k!((nk-1)-(k-1))!} = \frac{(nk-1)!n}{(k-1)!((nk-1)-(k-1))!} = n\binom{nk-1}{k-1}.$$

In other words, we have the following proposition.

Proposition 1. If $n > 0$ and $k \geq 0$ are integers then n divides $\binom{nk}{k}$.

It follows that $n! = 2 \cdot 3 \cdot \ldots \cdot n$ divides the number

$$\binom{2n}{n}\binom{3n}{n}\cdots\binom{n \cdot n}{n} = \frac{(2n)!}{n!n!}\frac{(3n)!}{(2n)!n!}\frac{(4n)!}{(3n)!n!}\cdots\frac{(n^2)!}{((n-1)n)!n!} = \frac{(n^2)!}{(n!)^n}.$$

Equivalently, we get the following result.

Proposition 2. If $n > 0$ is an integer then $(n!)^{n+1}$ divides $(n^2)!$.

By Proposition 2, to solve the problem it suffices to show that

$$\frac{(1^2 + 2^2 + \ldots + n^2)!}{(1^2)! \cdot (2^2)! \cdot (3^2)! \cdot \ldots \cdot (n^2)!}$$

is an integer. This follows from the following well known fact.

Proposition 3. For any non-negative integers $k_1, \ldots, k_s$ the number

$$\frac{(k_1 + \ldots + k_s)!}{k_1!k_2!\ldots k_s!}$$

is an integer.

Proposition 3 follows from the following straightforward observation:

$$\frac{(k_1 + \ldots + k_s)!}{k_1!k_2!\ldots k_s!} = \binom{k_1+k_2}{k_1}\binom{k_1+k_2+k_3}{k_1+k_2}\cdots\binom{k_1+k_2+\ldots+k_s}{k_1+k_2+\ldots+k_{s-1}}.$$

Problem. Show that for any non-negatime integers k, n the number $k!(n!)^k$ divides $(nk)!$.

Remark. The solution by Sasha Aksenchuk used the above problem to derive Proposition 2.