Problem 6. Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect at a point P. Let M, N be the midpoints of the sides AB and CD respectively. Prove that the area of the triangle PMN is equal to the quarter of the absolute value of the difference between the area of the triangle DAP and the area of the triangle BCP:

$$\text{area}(\triangle MNP) = \frac{1}{4}\left|\text{area}(\triangle DAP) - \text{area}(\triangle BCP)\right|.$$

Solution. Our first solution is based on the following basic fact from synthetic Euclidean geometry:

Theorem 1. The segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is equal to the half of the length of the third side.

Let K, L be the midpoints of the sides DA and BC respectively. By Theorem 1, KM and LN are parallel to BD and have length equal to half of the length of BD. Thus KM and LN are parallel and have the same length. Similarly, ML and NK are parallel and have the same length. In other words, the quadrilateral $KMLN$ is a parallelogram.

Let the line AC intersect the segments KM and LN at points S, U respectively. Then SU is parallel to ML and NK. Similarly, let the line BD intersect the segments ML and NK at points T, W respectively, so TW is parallel to KM and and LN. The point P is the intersection of segments SU and TW, hence it is inside the parallelogram $KMLN$. Looking at the triangle $\triangle DAP$ we see that K is the midpoint of AD and KS is parallel to DP. It follows from Theorem 1 that S is the midpoint of AP. Similarly, T is the midpoint of BP, U is the midpoint of CP and W is the midpoint of DP.

Looking again at the triangle $\triangle DAP$, Theorem 1 tells us that the four triangles $\triangle KAS$, $\triangle KSW$, $\triangle SPW$, $\triangle DKW$ are congruent. In particular, $2 \cdot \text{area}(KWPS) = \text{area}(APD)$. Similarly, looking at the triangle $\triangle BPC$, we see that $2 \cdot \text{area}(TPUL) = \text{area}(BPC)$. Thus we need to show that

$$2 \cdot \text{area}(\triangle MNP) = |\text{area}(KWPS) - \text{area}(TPUL)|.$$

We focus now on the parallelogram $KMLN$. Without loss of generality we may assume that P is in the triangle $\triangle KMN$ (if it is in the triangle $\triangle LMN$, the argument is the same). Since $KMLN$, $SMTP$, and $NUPW$ are parallelograms, we have

$$\text{area}(\triangle KMN) = \text{area}(\triangle MLN),\ \text{area}(\triangle SMP) = \text{area}(\triangle MPT) \text{ and } \text{area}(\triangle NPW) = \text{area}(\triangle NUP).$$

Note that

$$\text{area}(\triangle KMN) = \text{area}(KWPS) + \text{area}(\triangle SMP) + \text{area}(\triangle NPW) + \text{area}(\triangle MPN)$$

and

$$\text{area}(\triangle MLN) = \text{area}(TPUL) + \text{area}(\triangle MPT) + \text{area}(\triangle NUP) - \text{area}(\triangle MPN).$$

It follows that

$$\text{area}(KWPS) + \text{area}(\triangle MPN) = \text{area}(TPUL) - \text{area}(\triangle MPN),$$

i.e.

$$2 \cdot \text{area}(\triangle MNP) = \text{area}(TPUL) - \text{area}(KWPS) = |\text{area}(KWPS) - \text{area}(TPUL)|$$

as required.

Remark. Note that the special case of the result, that when P is on the line MN then the parallelograms $KWPS$ and $TPUL$ have equal areas, is exactly Proposition 43 of Book 1 of *Elements* by Euclid.

Second solution. Our second solution is similar to the solution submitted by Sasha Aksenchuk and uses basic analytic geometry. We use a coordinate system on the plane such that $P = (0, 0)$ $A = (0, a)$ and $C = (0, c)$ for some $a > 0 > c$ (so AC is the y-axis). The line BD has equation $y = tx$ for some t. We may assume that $B = (b, tb)$ and $D = (d, td)$ for some $b > 0 > d$. Then $M = (b/2, (a + tb)/2$ and $N = (d/2, (c + dt)/2$.

Recall now that if $E = (x_1, y_1)$, $F = (x_2, y_2)$ and $G = (x_3, y_3)$ are three points then

$$\text{area}(\triangle EFG) = \frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right|.$$

It follows that

$$\text{area}(\triangle PAD) = \frac{1}{2}|-da| = \frac{-ad}{2},$$

$$\text{area}(\triangle PCB) = \frac{1}{2}|-bc| = \frac{-bc}{2},$$

$$\text{area}(\triangle PMN) = \frac{1}{2}\left|\frac{b}{2}\frac{c+dt}{2} - \frac{d}{2}\frac{a+tb}{2}\right| = \frac{1}{2}\left|\frac{bc}{4} - \frac{ad}{4}\right| = \frac{1}{4}\left|\frac{-ad}{2} - \frac{-bc}{2}\right|.$$

Thus

$$\text{area}(\triangle MNP) = \frac{1}{4}\left|\text{area}(\triangle DAP) - \text{area}(\triangle BCP)\right|.$$