Problem 4. A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ has the following properties:

a) the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous on $\mathbb{R}^2$;

b) $\left(\dfrac{\partial f}{\partial x}(x,y)\right)^2 + \left(\dfrac{\partial f}{\partial y}(x,y)\right)^2 \le \dfrac{\partial f}{\partial x}(x,y)$ for every $(x,y) \in \mathbb{R}^2$;

c) $f(x,0) = 0$ for all $x \in \mathbb{R}$.

Prove that $f(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

Solution. Condition b) implies that $\dfrac{\partial f}{\partial x}(x,y) \ge \left(\dfrac{\partial f}{\partial x}(x,y)\right)^2$ for every $(x,y) \in \mathbb{R}^2$. It follows that

$$0 \le \frac{\partial f}{\partial x}(x,y) \le 1 \text{ for all } (x,y) \in \mathbb{R}^2. \tag{1}$$

The last inequality and condition b) yield

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \le 1 \text{ for all } (x,y) \in \mathbb{R}^2.$$

Suppose $y \neq 0$. By the mean value theorem we have

$$\frac{f(x,y) - f(x,0)}{y - 0} = \frac{\partial f}{\partial y}(x,u)$$

for some u between 0 and y. Since $f(x,0) = 0$ and $\left|\dfrac{\partial f}{\partial y}(x,u)\right| \le 1$, we conclude that

$$|f(x,y)| \le |y| \text{ for all } (x,y) \in \mathbb{R}^2. \tag{2}$$

Since $\dfrac{\partial f}{\partial x}(x,y) \ge 0$ for every $(x,y) \in \mathbb{R}^2$, for a fixed y the function $f(x,y)$ is a non decreasing function of x. In other words,

$$f(x_1,y) \le f(x_2,y) \quad \text{for all } x_1, x_2, y \in \mathbb{R}. \tag{3}$$

Fix $w > 0$. For $u_1 > u$ consider the double integral

$$\int_0^w \int_u^{u_1} \frac{\partial f}{\partial x}(x,y)\,dx\,dy = \int_0^w (f(u_1,y) - f(u,y))\,dy$$

From (2) we see that

$$f(u_1,y) - f(u,y) \le 2y$$

for $y > 0$. Thus

$$\int_0^w \int_u^{u_1} \frac{\partial f}{\partial x}(x,y)\,dx\,dy \le 2\int_0^w y\,dy = w^2 \tag{4}$$

On the other hand, by Fubini's Theorem, we have

$$\int_0^w \int_u^{u_1} \frac{\partial f}{\partial x}(x,y)\,dx\,dy = \int_u^{u_1} \int_0^w \frac{\partial f}{\partial x}(x,y)\,dy\,dx \ge \int_u^{u_1} \int_0^w \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy\,dx \tag{5}$$

(continuity of the partial derivative allows us to apply Fubini's Theorem). Recall now the Cauchy-Schwarz inequality:

$$\left(\int_a^b g(y)h(y)\,dy\right)^2 \le \left(\int_a^b g^2(y)\,dy\right)\left(\int_a^b h^2(y)\,dy\right).$$

Applying this inequality to $a = 0$, $b = w$, $g(y) = \dfrac{\partial f}{\partial y}(x,y)$ and $h(y) = 1$, we see that

$$\left(\int_0^w \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy\right)\int_0^w 1\,dy \ge \left(\int_0^w \frac{\partial f}{\partial y}(x,y)\,dy\right)^2 = (f(x,w) - f(x,0))^2 = f(x,w)^2,$$

i.e.

$$\int_0^w \left(\frac{\partial f}{\partial y}(x,y)\right)^2 dy \ge \frac{1}{w} f(x,w)^2.$$

This observation and (5) give us

$$\int_0^w \int_u^{u_1} \frac{\partial f}{\partial x}(x,y) dx dy \ge \frac{1}{w} \int_u^{u_1} f(x,w)^2 dx \tag{6}$$

Putting (4) and (6) together we see that

$$\int_u^{u_1} f(x,w)^2 dx \le w^3. \tag{7}$$

Suppose now that $f(a,w) \neq 0$ for some a. If $f(a,w) > 0$ then $f(x,w)^2 \ge f(a,w)^2$ for all $x \ge a$ by (3). It follows from (7) that

$$w^3 \ge \int_a^b f(x,w)^2 dx \ge (b-a) f(a,w)^2$$

for any $b > a$. Letting b increase to infinity, we get a contradiction ($w^3 \ge \infty$). Similarly, if $f(a,w) < 0$ then $f(x,w)^2 \ge f(a,w)^2$ for all $x \le a$ by (3). It follows from (7) that

$$w^3 \ge \int_b^a f(x,w)^2 dx \ge (a-b) f(a,w)^2$$

for any $b < a$. Letting b decrease to minus infinity, we again get a contradiction ($w^3 \ge \infty$). This proves that our assumption that $f(a,w) \neq 0$ must be false, i.e. $f(a,w) = 0$.

We proved that $f(a,w) = 0$ for every a and every $w > 0$. What about $w < 0$? We could adjust the above argument by considering integrals $\int_w^0$ instead of $\int_0^w$ and replacing w by $|w|$ where necessary. A more clever way is to observe that the function $f_1(x,y) = f(x,-y)$ also satisfies the conditions a),b),c) of the problem. Thus $f_1(a,w) = 0$ for all $w > 0$ and all a, i.e. $f(a,-w) = 0$ for all $w > 0$ and all a. This completes our proof that $f(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.