Problem 2. Let $p(x) = cx^n + c_1x^{n-1} + \dots$ be a polynomial of degree n with real coefficients and the leading coefficient $c \neq 0$. Prove that at least one of the numbers $|p(0)|, |p(1)|, \dots, |p(n)|$ is greater or equal than $\frac{|c|n!}{2^n}$. Prove furthermore that this bound is best possible.

Solution. For any function $f : \mathbb{R} \longrightarrow \mathbb{R}$ define a new function Δf by the formula

$$\Delta f(x) = f(x+1) - f(x).$$

Recall now the following simple fact from calculus:

a function $p(x)$ is a polynomial of degree $n \geq 1$ with leading coefficient c if and only if the derivative $p'(x)$ is a polynomial of degree $n-1$ with leading coefficient nc.

Using this fact, we prove the following key observation needed for our solution.

Lemma. *Let $p(x)$ be a polynomial of degree n with the leading coefficient c. Then Δp is a polynomial of degree $n-1$ with the leading coefficient nc.*

We prove the Lemma by induction on n. When $n = 1$ then $p(x) = cx + d$ is linear and $\Delta p(x) = c(x+1) + d - (cx + d) = c$, so the claim is true for $n = 1$. Suppose that $n > 1$ and the result is true for polynomials of degree $n-1$. If $p(x)$ is a polynomial of degree n with the leading coefficient c then $p'(x)$ is a polynomial of degree $n-1$ with the leading coefficient nc. Then, by the inductive assumption, $\Delta p'(x)$ is a polynomial of degree $n-2$ with the leading coefficient $n(n-1)c$. Note that

$$\Delta p'(x) = p'(x+1) - p'(x) = (p(x+1) - p(x))' = (\Delta p(x))'$$

so $(\Delta p(x))'$ is a polynomial of degree $n-2$ with the leading coefficient $(n-1)nc$. It follows that Δp is a polynomial of degree $n-1$ with the leading coefficient nc. Thus the Lemma is true for $p(x)$. In other words, we showed that the Lemma is true for all polynomials of degree n. By the method of induction, the Lemma is true for all n.

Remark. A more straightforward argument is to use the binomial formula to see that

$$\Delta x^n = (x+1)^n - x^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k}$$

is a polynomial of degree $n-1$ with the leading coefficient n.

We are ready now to prove the first part of our problem by induction on n. If $n = 0$ then $p(x) = c$ is a constant polynomial and the result is obvious (we use the convention $0! = 1$). Suppose $n \geq 1$ and the result holds for polynomials of degree $n-1$. Let $p(x)$ be a polynomial of degree n with the leading coefficient c. Then $\Delta p(x)$ is a polynomial of degree $n-1$ and the leading coefficient nc. Be the inductive assumption, there is integer k such that $0 \leq k \leq n-1$ and $|\Delta p(k)| \geq nc(n-1)!/2^{n-1}$. In other words,

$$\frac{cn!}{2^{n-1}} \leq |p(k+1) - p(k)| \leq |p(k+1)| + |p(k)| \leq 2\max(|p(k+1)|, |p(k)|).$$

Thus

$$\max(|p(k+1)|, |p(k)|) \geq \frac{cn!}{2^n}.$$

In other words, the larger of the two values $|p(k)|$ and $|p(k+1)|$ must be at least $cn!/2^n$, so the result is true for $p(x)$. In summary, we showed that the first part of the problem is true for all polynomials of degree n. By the method of induction, the first part of the problem is true for all n.

In order to show that the bound is best possible, we need to find a polynomial of degree n with the leading coefficient c such that $|p(k)| \leq cn!/2^n$ for $k = 0, 1, \dots, n$. Suppose that for every n we can construct a polynomial $q_n(x)$ of degree n such that $q_n(k) = (-1)^k$ for $k = 0, 1, \dots, n$. Clearly there is at most one such polynomial for each n (if two polynomials of degree n assume the same values at $n+1$ different arguments then they are equal). Note that for $k = 0, 1, \dots n-1$ we have

$$\Delta q_n(k) = (-1)^{k+1} - (-1)^k = -2(-1)^k = -2q_{n-1}(k).$$

Since both q_{n-1} and Δq_n are polynomials of degree $n-1$, we conclude that $\Delta q_n(x) = -2q_{n-1}(x)$. Let c_n be the leading coefficient of q_n. Then $c_0 = 1$ and $nc_n = -2c_{n-1}$. It is now easy to see that $c_n = (-2)^n/n!$ (use a straightforward induction). We see that

$$|q_n(k)| = 1 = \frac{|c_n|n!}{2^n}$$

for $k = 0, 1, \ldots, n$. This means that the bound in the problem is indeed best possible.

It remains to show that polynomials q_n exist. Clearly $q_0 = 1$ works. Suppose that q_{n-1} exists . If q_n exists, then $q_n(k) - q_{n-1}(k) = 0$ for $k = 0, 1, \ldots, n-1$. Since $q_n - q_{n-1}$ is a polynomial of degree n, we must have $q_n(x) - q_{n-1}(x) = ax(x-1)\ldots(x-(n-1))$ for some constant a. Thus q_n must be of the form $f_a(x) = q_{n-1}(x) + ax(x-1)\ldots(x-(n-1))$. Clearly $f_a(k) = q_{n-1}(k) = (-1)^k$ for $k = 0, 1, \ldots, n-1$. So we need a such that $f_a(n) = (-1)^n$. Since $f_a(n) = q_{n-1}(n) + an!$, we easily see that $a = ((-1)^n - q_{n-1}(n))/n!$ works. Thus $q_n(x)$ indeed exists. This completes our solution.

Remark. Mithun in his solution constructs the polynomials q_n using Lagrange's interpolation formula, which says that given distinct numbers $x_0, x_1, \ldots, x_n$ and any numbers $a_0, \ldots, a_n$ the polynomial

$$L(x) = \sum_{j=0}^{n} a_j \frac{\prod_{i\neq j}(x - x_i)}{\prod_{i\neq j}(x_j - x_i)}$$

has degree at most n and satisfies $L(x_i) = a_i$ for $i = 0, 1, \ldots, n$.

The operation Δ is a very useful tool to study polynomials. We suggest the following problems related to Δ.

Problem. Let f be a polynomial of degree n such that the value $f(k)$ is an integer at $n+1$ consecutive integers k. Show that $f(k)$ is an integer for all integers k.

Problem. Suppose that a polynomial $p(x)$ of degree n with the leading coefficient c satisfies $|p(k)| \leq |c|n!/2^n$ for $k = 0, 1, \ldots, n$. Prove that $p(x) = dq_n(x)$ for some constant d.

Problem. Define the polynomial $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$, where $\binom{x}{0} = 1$ and $\binom{x}{1} = x$.

a) Prove that $\Delta\binom{x}{k} = \binom{x}{k-1}$.

b) Prove that every polynomial of degree n can be expressed as $\sum_{k=0}^{n} c_k \binom{x}{k}$ for unique numbers $c_0, c_1, \ldots, c_n$.

c) Suppose that $p(x) = \sum_{k=0}^{n} c_k \binom{x}{k}$. Prove that $p(k)$ is an integer for all integers k if and only if each c_i is an integer.

d) Prove that $q_n(x) = \sum_{k=0}^{n} (-2)^k \binom{x}{k}$.