Problem 2. Let $d(n)$ be the smallest number such that among any $d(n)$ points inside a regular n-gon with side of length 1 there are two points whose distance from each other is at most 1. Prove that

(a) $d(n) = n$ for $4 \le n \le 6$.

(b) $\lim_{n\to\infty} \frac{d(n)}{n} = \infty$.

Solution. We start with some elementary observations from plane geometry. Let $P_1 \dots P_n$ be a regular n-gon with side of length 1, let O be the center of the n-gon, and let $R_n = OP_1$ be the radius of the circle circumscribed on the n-gon. The triangle P_1OP_2 is isosceles and $\angle P_1OP_2 = 2\pi/n$. It follows that

$$OP_1 = R_n = \frac{1}{2\sin(\frac{\pi}{n})}.$$

Note that $R_n \le 1$ for $n \le 6$ and $\lim_{n\to\infty} R_n = \infty$.

a) Suppose that $n \le 6$ and $Q_1, \dots, Q_n$ are points inside the n-gon. We will show that there are two among the points which are at distance at most 1. If one of these points, say Q_i, is O, then $Q_iQ_j = OQ_j \le R_n \le 1$. So we may assume that none of our points is O. Furthermore, we may assume that the points are numbered so that when we rotate the ray OQ_1 counterclockwise we will encounter points $Q_2, ..., Q_n$ in that order. We see that one of the angles $\angle Q_1OQ_2$, $\angle Q_2OQ_3$, ..., $\angle Q_nOQ_1$ does not exceed $2\pi/n$. Suppose that $\alpha = \angle Q_iOQ_j \le 2\pi/n$. Since $OQ_k \le R_n$ for every k, extending the sides OQ_i and OQ_j of the triangle OQ_iQ_j, we see that triangle OQ_iQ_j is contained in a triangle OAB such that $OA = OB = R_n$ and $\angle AOB = \alpha$. It follows that

$$AB = 2AO\sin(\alpha/2) \le 2R_n\sin(\pi/n) = 1.$$

We see that all sides of the triangle OAB have length smaller or equal than 1. Thus the same is true for the sides of the triangle Q_iOQ_j (which is contained in the triangle OAB). In particular, $Q_iQ_j \le 1$. This proves that $d(n) \le n$ for $n \le 6$.

We leave it as a simple exercise to find 3 points in a square, 4 points in a regular pentagon, and 5 points in a regular hexagon, such that the distance between any two of them is bigger than 1. It follows that $d(n) = n$ for $n = 4, 5, 6$.

b) We will show that for any positive integer k we have $d(n) > kn$ for all n sufficiently large. Fix k and $t > 1$. Since

$$\lim_{n\to\infty}\left(\cos\left(\frac{\pi}{n}\right) - 2kt\sin\left(\frac{2\pi}{n}\right)\right) = 1$$

the inequality

$$\cos\left(\frac{\pi}{n}\right) - 2kt\sin\left(\frac{2\pi}{n}\right) > \frac{1}{\sqrt{2}}$$

holds for all sufficiently large n, say for all $n > N(k)$. Let $n > N(k)$. For each odd integer $i < n$ and $j = 0, 1, \dots 2k$ choose points $Q_{i,j}$ on the segment OP_i such that $P_iQ_{i,j} = jt$. Then

$$OQ_{i,j} \ge OQ_{i,2k} = R_n - P_iQ_{i,2k} = \frac{1}{2\sin(\frac{\pi}{n})} - 2kt = \frac{1}{\sin(\frac{2\pi}{n})}\left(\cos\left(\frac{\pi}{n}\right) - 2kt\sin\left(\frac{2\pi}{n}\right)\right) > \frac{1}{\sqrt{2}\sin(\frac{2\pi}{n})}.$$

We constructed $(2k+1)\left\lfloor\frac{n}{2}\right\rfloor$ points $Q_{i,j}$ inside the n-gon. Consider any two points Q_{i_1,j_1}, Q_{i_2,j_2} among them. If $i_1 = i_2$ then

$$Q_{i_1,j_1}Q_{i_2,j_2} = |j_1 - j_2|t > 1.$$

If $i_1 \ne i_2$ then the angle $\alpha = \angle Q_{i_1,j_1}OQ_{i_2,j_2}$ satisfies

$$\frac{4\pi}{n} \le \alpha = \angle P_{i_1}OP_{i_2}.$$

The law of cosines for the triangle $Q_{i_1,j_1}OQ_{i_2,j_2}$ yields

$$(Q_{i_1,j_1}Q_{i_2,j_2})^2 = (OQ_{i_1,j_1})^2 + (OQ_{i_2,j_2})^2 - 2(OQ_{i_1,j_1})(OQ_{i_2,j_2})\cos\alpha =$$

$$(OQ_{i_1,j_1} - OQ_{i_2,j_2})^2 + 2(OQ_{i_1,j_1})(OQ_{i_2,j_2})\sin^2\left(\frac{\alpha}{2}\right) > 2\left(\frac{1}{\sqrt{2}\sin(\frac{2\pi}{n})}\right)^2 \sin^2\left(\frac{\alpha}{2}\right) = \frac{\sin^2\left(\frac{\alpha}{2}\right)}{\sin(\frac{2\pi}{n})} \geq 1.$$

Thus the distance between any two of our points is larger than 1. It follows that

$$d(n) > (2k+1)\left\lfloor \frac{n}{2} \right\rfloor \geq (2k+1)(n-1)/2 \geq kn$$

as long as $n > N(k)$ and $n > 2k+1$.

Problem. Prove that if $n > 6$ then $d(n) > n$.

Problem. Prove that there exists positive numbers a, b such that

$$an^2 < d(n) < bn^2$$

for all n.

We do not know the answer to the following question.

Question. Does the sequence $\frac{d(n)}{n^2}$ converge?