Problem 7. Let ABC be an equilateral triangle and P any point inside ABC. Show that the segments AP, BP, CP are sides of some triangle $T(P)$ and find P for which the area of $T(P)$ is largest.

Solution. Recall that there is a triangle with sides of length a, b, c (where a, b, c are positive real numbers) if and only if $a + b > c$, $a + c > b$, and $b + c > a$. Let $AP = a$, $BP = b$, $CP = c$. We will show that $b + c > a$ (the other two inequalities are proved in the same way). Since BPC is a triangles, we have $b + c > BC$. Let X be the intersection of the lines AP and BC. One of the angles $\angle AXB$, $\angle AXC$ is greater or equal than $90°$, say $\angle AXB \geq 90°$. Recall that in any triangle, across a bigger angle is a bigger side. Since $\angle AXB$ is the biggest angle of the triangle AXB, AB is the biggest side of this triangle. Thus $a = AP < AX < AB = BC < b + c$.

We showed that there is a triangle $T(P)$ with sides of length a, b, c. In order to discuss the area of $T(P)$ we need to recall Heron's formula:

Heron's formula. The area S of a triangle with sides of length a, b, c is equal to $S = \sqrt{p(p-a)(p-b)(p-c)}$, where $p = (a + b + c)/2$.

From Heron's formula we get

$$16S^2 = (a+b+c)(a+b-c)(c+b-a)(c+a-b) = ((a+b)^2 - c^2)(c^2 - (a-b)^2) =$$
$$(a+b)^2c^2 + (a-b)^2c^2 - (a+b)^2(a-b)^2 - c^4 = c^2((a+b)^2 + (a-b)^2) - (a^2-b^2)^2 - c^4 =$$
$$c^2(2a^2+2b^2) + 2a^2b^2 - a^4 - b^4 - c^4 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 = 4b^2c^2 - (b^2+c^2-a^2)^2.$$

We now need to compute the squares a^2, b^2, c^2. There are various possibilities here. We choose to use a coordinate system in which the x-axis is the line BC and the y-axis is the perpendicular bisector of the segment BC (Slava Kargin in his solution placed the triangle in 3-space so that its vertices are (1,0,0), (0,1,0), (0,0,1)). Thus the vertices A, B, C have coordinates $A(0, u\sqrt{3})$, $B(-u, 0)$ $C(u, 0)$, where $2u$ is the length of the sides of the triangle ABC. If P has coordinates (x, y) then

$$a^2 = x^2 + (y - u\sqrt{3})^2, \quad b^2 = (x+u)^2 + y^2, \quad c^2 = (x-u)^2 + y^2.$$

It follows that

$$16S^2 = 4b^2c^2 - (b^2+c^2-a^2)^2 = 4((x+u)^2+y^2)((x-u)^2+y^2) - ((x+u)^2+y^2+(x-u)^2+y^2-x^2-(y-u\sqrt{3})^2)^2 =$$
$$4(x^2+y^2+u^2+2xu)(x^2+y^2+u^2-2xu) - (x^2-u^2+y^2+2\sqrt{3}yu)^2 =$$
$$4((x^2+y^2+u^2)^2 - 4x^2u^2) - (x^4 + 2x^2(y^2+2\sqrt{3}yu-u^2) + (y^2+2\sqrt{3}yu-u^2)^2) =$$
$$4(x^4 + 2x^2(y^2-u^2) + (u^2+y^2)^2) - (x^4 + 2x^2(y^2+2\sqrt{3}yu-u^2)) - (y^2+2\sqrt{3}yu-u^2)^2 =$$
$$3x^4 + 2x^2(4y^2-4u^2-y^2-2\sqrt{3}yu+u^2) + 4(u^2+y^2)^2 - (y^2+2\sqrt{3}yu-u^2)^2 =$$
$$3x^4 + 2x^2(3y^2-2\sqrt{3}yu-3u^2) + (3y^4 - 4\sqrt{3}y^3u - 2y^2u^2 + 4\sqrt{3}yu^3 + 3u^4) =$$
$$3x^4 + 2\sqrt{3}x^2(\sqrt{3}y^2-2yu-\sqrt{3}u^2) + (\sqrt{3}y^2-2yu-\sqrt{3}u^2)^2 - (\sqrt{3}y^2-2yu-\sqrt{3}u^2)^2 + (3y^4-4\sqrt{3}y^3u-2y^2u^2+4\sqrt{3}yu^3+3u^4) =$$
$$\left(\sqrt{3}x^2+\sqrt{3}y^2-2yu-\sqrt{3}u^2\right)^2 - (3y^4+4y^2u^2+3u^4-4\sqrt{3}y^2u+4\sqrt{3}yu^3-6u^2y^2) + (3y^4-4\sqrt{3}y^3u-2y^2u^2+4\sqrt{3}yu^3+3u^4) =$$
$$\left(\sqrt{3}x^2+\sqrt{3}y^2-2yu-\sqrt{3}u^2\right)^2 = 3\left(x^2 + \left(y - \frac{u}{\sqrt{3}}\right)^2 - \frac{4}{3}u^2\right)^2$$

(the idea behind the above manipulations was to express the formula as a quadratic polynomial in x^2 and then complete to squares). We conclude that

$$S = \frac{\sqrt{3}}{4}\left|x^2 + \left(y - \frac{u}{\sqrt{3}}\right)^2 - \frac{4}{3}u^2\right|.$$

Note now that the equation

$$x^2 + \left(y - \frac{u}{\sqrt{3}}\right)^2 = \frac{4}{3}u^2$$

describes the circle circumscribed about the triangle ABC and for every point (x, y) inside the triangle we have $x^2 + \left(y - \frac{u}{\sqrt{3}}\right)^2 < \frac{4}{3}u^2$. It follows that S is largest when $x^2 + \left(y - \frac{u}{\sqrt{3}}\right)^2 = 0$, i.e. when $P = (0, u/\sqrt{3})$ is the cicrumcenter of the triangle ABC.

Problem 1. Show that non-negative numbers a, b, c are lengths of sides of some triangle if and only if $(a+b+c)(a+b-c)(a+c-b)(b+c-a) > 0$. Conclude that for any point P on the plane which is not on the circle circumscribed about ABC there is a triangle with sides AP, BP, CP.

Problem 2. Our solution shows that P is on the circumcirle of the triangle ABC if and only if one of the lengths AP, BP, CP is the sum of the other two. Find a more direct proof of this observation.

Problem 3. Find all points P for which the perimeter of the triangle $T(P)$ is smallest possible.