**Problem 1.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such for any $x, y$ either $f(x-y)+f(y)=f(x)$ or $f(y-x)+f(x)=f(y)$. Prove that the sum $f(x)+f(-x)$ assumes at most two different values. Find an example when $f(x)+f(-x)$ assumes two different values.

**Solution.** Let $a, b \in \mathbb{R}$. Taking $x=-a$, $y=b$ we see that

$$f(-a-b)+f(b)=f(-a) \ \text{ or } \ f(a+b)+f(-a)=f(b).$$

Taking $x=a$, $y=-b$ we see that

$$f(a+b)+f(-b)=f(a) \ \text{ or } \ f(-a-b)+f(a)=f(-b).$$

This leads to three cases: $f(a+b)+f(-a)=f(b)$, or $f(a+b)+f(-b)=f(a)$, or $f(-a-b)+f(b)=f(-a)$ and $f(-a-b)+f(a)=f(-b)$.

Suppose that $f(a+b)+f(-a)=f(b)$. Taking $x=a+b$ and $y=a$ we see that either $f(b)+f(a)=f(a+b)$ or $f(-b)+f(a+b)=f(a)$. In the former case, we get $f(a)+f(-a)=0$. In the latter case, $f(a)+f(-a)=f(b)+f(-b)$.

Similarly, suppose that $f(a+b)+f(-b)=f(a)$. Taking $x=a+b$ and $y=b$ we see that either $f(a)+f(b)=f(a+b)$ or $f(-a)+f(a+b)=f(b)$. In the former case, we get $f(b)+f(-b)=0$. In the latter case, $f(a)+f(-a)=f(b)+f(-b)$.

Finally, if $f(-a-b)+f(b)=f(-a)$ and $f(-a-b)+f(a)=f(-b)$ then $f(a)+f(-a)=f(b)+f(-b)$.

We showed that either $f(a)+f(-a)=f(b)+f(-b)$ or one of the numbers $f(a)+f(-a)$, $f(b)+f(-b)$ is 0. Since $a, b$ were arbitrary, this means that $f(x)+f(-x)$ assumes at most one non-zero value. This shows the first part of the problem.

Consider now the function $f(x)=\lfloor x \rfloor$. Here $\lfloor x \rfloor$ is the floor of $x$, i.e. the largest integer smaller or equal than $x$. Write $x=m+u$ and $y=n+w$, where $m, n$ are integers and $u, w \in [0,1)$. Then $f(x)=m$ and $f(y)=n$. If $u \geq w$ then $x-y=m-n+u-w$ and since $u-w \in [0,1)$, we see that $f(x-y)=m-n$. It follows that $f(x-y)+f(y)=f(x)$. If $u<w$ then $y-x=n-m+w-u$ and $w-u \in [0,1)$, so $f(y-x)=n-m$. Thus $f(y-x)+f(x)=f(y)$. This proves that $f$ satisfies the conditions of the problem. Clearly $f(x)+f(-x)=0$ when $x$ is an integer and $f(x)+f(-x)=-1$ if $x$ is not an integer.

**Another example (found by Levi Axelrod).** Consider the function

$$f(x)=\begin{cases} x & \text{if } x \leq 0 \\ 1+x & \text{if } x>0. \end{cases}$$

If both $x, y$ are positive and $x \geq y$ then $f(y-x)+f(x)=y-x+1+x=1+y=f(y)$.
If both $x, y$ are non-positive and $x \geq y$ then $f(y-x)+f(x)=y-x+x=y=f(y)$
If $x$ is positive and $y \leq 0$ then $f(x-y)+f(y)=1+x-y+y=1+x=f(x)$.
Thus $f$ satisfies the condition of the problem. Clearly $f(x)+f(-x)=1$ if $x \neq 0$ and $f(0)+f(-0)=0$.

Here are additional problems to work on.

**Problem.** Let $f$ be a function as in the problem.

a) Prove that the set $\{x : f(x)+f(-x)=0\}$ contains 0 and is closed under subtraction (i.e. it is a subgroup of $\mathbb{R}$ under addition).

b) Show that the function $h(x)=f(x)+f(-x)$ has the following property: if $h(x) \neq h(y)$ then $h(x-y)=h(x)-h(y)$ or $h(x-y)=h(y)-h(x)$.

**Problem.** Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $h(x)=h(-x)$ for all $x$ and for any $x, y$ such that $h(x) \neq h(y)$, either $h(x-y)=h(x)-h(y)$ or $h(x-y)=h(y)-h(x)$. Prove that $h$ assumes at most 3 different values. Find an example of such $h$ which assumes 3 different values.