Problem 1. Consider any collection of 2023 lines on a plane such that no two lines are parallel and no 3 lines share a common point. These lines divide the plane into some number of pieces. Show that at least 1348 of these pieces are triangles.

Solution. Let P be the set of all points of intersection of our lines and let T be the set of all triangular pieces among the pieces into which our lines divide the plane. Thus we need to show that T has at least 1348 elements.

Recall that a line on a plane divides the plane into 2 parts (open half-planes) called the sides of the line (the line itself does not belong to any of its sides). We have $2 \cdot 2023 = 4046$ pairs (l, H), where l is one of our lines and H is a side of l. Consider one such pair (l, H) and assume that H contains at least one point form P. Choose among the points of P in H a point A which is closest to l. There are two lines l_1, l_2 among our lines which pass through the point A. Let l_1 intersect l at B and l_2 intersect l at C. Then the triangle ABC is in T. Indeed, otherwise one of our 2023 lines would have a point inside the triangle ABC, hence it would intersect either the segments AB or the segment AC. The point of intersection would then be a point in P which is closer to l than A, contrary to our choice of A.

Thus, to every pair (l, H) such that H contains a point form P we associate a triangle in T whose one side is in l and one vertex is in H. A given triangle in T is associated to at most three pairs (l, H). It follows that $|T| \geq N/3$, where N is the number of pairs (l, H) such that l is one of our 2023 lines and H is a side of l which contains a point in P.

We claim that there are at most 2 pairs (l, H) such that H has no points in P. If true, this means that $N \geq 4046 - 2 = 4044$ and $|T| \geq 4044/3 = 1348$, as required. To justify our claim, note that for each l at least one side of l has points in P. Consider now any three of our lines l_1, l_2, l_3. Let l be a fourth of our lines and let A_i be the intersection of l and l_i, $i = 1, 2, 3$. Suppose that A_2 lies between A_1 and A_3 on the line l. Then the points A_1 and A_3 are in P and belong to opposite sides of l_2. This proves that among any three lines at least one has a point from P of each of its sides. Thus there are at most two of our lines which do not have a point from P on both of their sides, as claimed.

We leave the reader with the following two classical problems.

Problem. Consider any collection of n lines on a plane such that no two lines are parallel and no 3 lines share a common point. These lines divide the plane into $p(n)$ pieces. Find an explicit formula for $p(n)$ (it is a polynomial in n).

Problem. What is the largest number of pieces one can divide the interior of a circle into by choosing n points on the circle and joining any two of them by a line?