Problem 7. Find the smallest positive integer which cannot be expressed as a sum of 2023 or fewer Fibonacci numbers (not necessarily distinct). Recall that the Fibonacci numbers f_n are defined recursively as follows: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for all $n > 2$.

Solution. We start with stating some properties of the Fibonacci numbers.

1. $f_1 = f_2$.
2. $2f_2 = f_3$.
3. $2f_n = f_{n-2} + f_{n+1}$ for all $n \geq 3$.
4. $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$ for every $n \geq 1$.

Properties 1 and 2 are clear, property 3 is justified by

$$2f_n = f_n + f_n = f_{n-2} + f_{n-1} + f_n = f_{n-2} + f_{n+1},$$

and property 4 can be justified by a straightforward induction.

Proposition 1 *Let $n \geq 2$. Every positive integer smaller than $f_{2n+1} - 1$ is a sum of fewer than n distinct Fibonacci numbers.*

We prove Proposition 1 by induction on n. It is clear that Proposition 1 is true for $n = 2$. Suppose it is true for some $n \geq 2$. Consider an integer $K < f_{2(n+1)+1} - 1$.

If $K < f_{2n+1} - 1$ then the K is a sum of fewer than n Fibonacci numbers by the inductive assumption.

If $K = f_{2n+1} - 1$, then $K - f_{2n} = f_{2n-1} - 1 < f_{2n+1} - 1$ is a sum of fewer than n distinct Fibonacci numbers, each of which is less than f_{2n}, so K is a sum of fewer than $n + 1$ distinct Fibonacci numbers.

If $K = f_{2n+1}$, then K is a Fibonacci number, hence it is a sum of fewer that $n + 1$ Fibonacci numbers.

If $f_{2n+1} < K \leq f_{2n+2}$, then $0 < K - f_{2n+1} \leq f_{2n+2} - f_{2n+1} = f_{2n} < f_{2n+1} - 1$, so $K - f_{2n+1}$ is a sum of fewer than n distinct Fibonacci numbers each of which is less than f_{2n+1}, hence K is a sum of fewer than $n + 1$ distinct Fibonacci numbers.

If $f_{2n+2} < K \leq f_{2n+3} - 1$, then $0 < K - f_{2n+2} < f_{2n+3} - 1 - f_{2n+2} = f_{2n+1} - 1$, so $K - f_{2n+2}$ is a sum of fewer than n distinct Fibonacci numbers each of which is less than f_{2n+2}, hence K is a sum of fewer than $n + 1$ distinct Fibonacci numbers.

Putting the above observations together we see that Proposition 1 is true for $n + 1$. By the method of induction, Proposition 1 is true for all $n \geq 2$.

Proposition 2 *If $n \geq 2$ then $f_{2n+1} - 1$ is not a a sum of fewer than n distinct Fibonacci numbers.*

We prove Proposition 2 by induction on n. It is clear that $f_5 - 1 = 4$ is not a Fibonacci number, so Proposition 2 is true for $n = 2$. Suppose that Proposition 2 is true for some $n \geq 2$. If $f_{2(n+1)+1} - 1$ were a sum of fewer than $n+1$ distinct Fibonacci numbers then we would have integers $0 < m_1 < m_2 < \ldots < m_s$ such that $s < n+1$ and $f_{2n+3} - 1 = f_{m_1} + \ldots + f_{m_s}$. Since $f_1 + f_2 + \ldots + f_{2n+1} = f_{2n+3} - 1$ by property 4, we must have $m_s = 2n + 2$. Thus $f_{m_1} + \ldots + f_{m_{s-1}} = f_{2n+3} - 1 - f_{2n+2} = f_{2n+1} - 1$. In other words, $f_{2n+1} - 1$ would be a sum of fewer than n distinct Fibonacci numbers, contrary to our inductive assumption. It follows that $f_{2n+3} - 1$ is not a sum of fewer than $n + 1$ distinct Fibonacci numbers, i.e. Proposition 2 holds for $n + 1$. Thus Proposition 2 is true for all $n \geq 2$ by the method of mathematical induction.

Proposition 3 *Let $n \geq 2$. If a positive integer is a sum of fewer than n Fibonacci numbers then it is a sum of fewer than n distinct Fibonacci numbers.*

Consider a positive integer K which is a sum of fewer than n Fibonacci numbers. Thus $K = \sum m_j f_j$ for some non-negative integers m_j such that $\sum m_j < n$. Among all such expressions for K choose one for which the sum $\sum m_j 3^j$ is largest possible. We will show that each m_j in this expression for K is either 0 or 1, which proves Proposition 3. Suppose that $m_i \geq 2$ for some i.

If $i = 1$, then using property 1 we have $K = (m_1 - 2)f_1 + (m_2 + 2)f_2 + \sum_{j>2} m_j f_j$, $(m_1 - 2) + (m_2 + 2) + \sum_{j>2} m_j = \sum m_j < n$, and $(m_1 - 2) \cdot 3 + (m_2 + 2) \cdot 3^2 + \sum_{j>2} m_j 3^j > \sum m_j 3^j$, contrary to our choice of the numbers m_j.

If $i = 2$, then by property 2 we have $K = m_1 f_1 + (m_2 - 2) f_2 + (m_3 + 1) f_3 + \sum_{j>3} m_j f_j$, $m_1 + (m_2 - 2) + (m_3 + 1) + \sum_{j>3} m_j = -1 + \sum m_j < n$, and $m_1 \cdot 3 + (m_2 - 2) \cdot 3^2 + (m_3 + 1) \cdot 3^3 + \sum_{j>3} m_j 3^j > \sum m_j 3^j$, contrary to our choice of the numbers m_j.

If $i \geq 3$, then using property 3 we have

$$K = \sum_{j<i-2} m_j f_j + (m_{i-2} + 1) f_{i-2} + m_{i-1} f_{i-1} + (m_i - 2) f_i + (m_{i+1} + 1) f_{i+1} + \sum_{j>i+1} m_j f_j,$$

$$\sum_{j<i-2} m_j + (m_{i-2} + 1) + m_{i-1} + (m_i - 2) + (m_{i+1} + 1) + \sum_{j>i+1} m_j = \sum m_j < n,$$

and

$$\sum_{j<i-2} m_j 3^j + (m_{i-2} + 1) 3^{i-2} + m_{i-1} 3^{i-1} + (m_i - 2) 3^i + (m_{i+1} + 1) 3^{i+1} + \sum_{j>i+1} m_j 3^j > \sum m_j 3^j,$$

contrary to our choice of the numbers m_j.

In each case we arrived at a contradiction, which shows that we must have $m_j < 2$ for all j.

Proposition 1 tells us that every integer smaller than $f_{2 \cdot 2024+1} - 1$ is a sum of fewer than 2024 distinct Fibonacci numbers. Proposition 2 says that $f_{2 \cdot 2024+1} - 1$ is not a sum of fewer than 2024 distinct Fibonacci numbers and Proposition 3 implies then that $f_{2 \cdot 2024+1} - 1$ is not a sum of fewer than 2024 Fibonacci numbers. Thus $f_{4049} - 1$ is the smallest positive integer which is not a sum of fewer than 2024 Fibonacci numbers.