Problem 6. Find all bounded continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy the following condition:

$$f^2(x) - f^2(y) = f(x+y)f(x-y) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Here $f^2 = f \cdot f$ is the square of the function f (and not the composition of f with itself).

Solution. Taking $x = y = 0$ in (1) we get $f(0) = 0$. Taking $x = -y$, we get $f^2(y) = f^2(-y)$, so if $f(y) = 0$ then also $f(-y) = 0$. Taking $x = 0$ we get $-f^2(y) = f(y)f(-y)$. Thus if $f(y) \neq 0$ then $f(-y) = -f(y)$. The last two observations combined mean that $f(-y) = -f(y)$ for all y, i.e. f is an odd function. Taking $x = 2y$ in (1) yields $f^2(2y) - f^2(y) = f(3y)f(y)$. It follows that if $f(y) = 0$ then $f(2y) = 0$. From (1) we get also the following equality:

$$f^2(x+y) - f^2(x-y) = f(2x)f(2y) \quad \text{for all } x, y \in \mathbb{R}. \tag{2}$$

Suppose now that $f(y) = 0$ for some y. Then from (1) and (2) we get

$$f^2(x) = f(x+y)f(x-y) \text{ and } f^2(x+y) = f^2(x-y) \quad \text{for all } x \in \mathbb{R}.$$

We see that $f(x+y) = \pm f(x-y)$. Since $f(x+y)f(x-y) = f^2(x) \geq 0$, we have $f(x+y) = f(x-y)$ for all $x \in \mathbb{R}$. Equivalently, $f(x+2y) = f(x)$ for all $x \in \mathbb{R}$. Furthermore, if also $f(x) = 0$, then we have $f(x+y) = f(x-y) = 0$.

Let $Z = \{a : f(a) = 0\}$ be the set of all zeros of the function f. We summarize the above discussion as follows:

- Z1. $0 \in Z$.
- Z2. if $x, y \in Z$ then $x + y \in Z$ and $x - y \in Z$.
- Z3. if $y \in Z$ then $f(x+2y) = f(x)$ for all $x \in \mathbb{R}$.
- Z4. if $y \in Z$ then $f^2(x+y) = f^2(x)$ for all $x \in \mathbb{R}$.

Suppose now that $Z = \{0\}$. Since f is continuous, it satisfies the intermediate value property. It follows that on $(0, \infty)$ the function f is either positive or negative. Replacing f by $-f$ if necessary, we may assume that $f(x) > 0$ for all $x > 0$ (note that if f satisfies (1) the so does $Af(bx)$ for any real numbers A and b). Thus, if $x > y > 0$ then $f^2(x) - f^2(y) = f(x+y)f(x-y) > 0$, so $f(x) > f(y)$. In other words, f is increasing on $(0, \infty)$. Since f is bounded, the limit $\lim_{x \to \infty} f(x) = L$ exists and it is positive. From (1) we have $f^2(2x) - f^2(x) = f(3x)f(x)$. By passing to the limit when $x \to \infty$, we get $L^2 - L^2 = L^2$, i.e. $L = 0$, a contradiction. We see that the assumption that f is bounded implies that Z must have non-zero elements. By Z1 and Z2, Z has positive elements.

Suppose that Z has the smallest positive element a. If b is any real number, then $na \leq b < (n+1)a$ for some integer n. Thus $0 \leq b - na < a$. Now if $b \in Z$ then by Z2 both na and $b - na$ are in Z. Since a is the smallest positive element in Z, we must have $b - na = 0$, i.e. $b = na$. In other words, if Z has the smallest positive element a then $Z = \{na : n \in \mathbb{Z}\}$.

Suppose now that Z does not have the smallest positive element. Take a_1 to be a positive element in Z. Then Z has a positive element a_2 which is smaller that a_1. Repeating this reasoning, we construct a decreasing sequence $a_1 > a_2 > \ldots$ of positive elements in Z. Any decreasing sequence of positive numbers converges, so $\lim_{n\to\infty} a_n$ exists. It follows that the sequence $b_n = a_{n+1} - a_n$ converges to 0. By Z2, every b_n is in Z. Thus Z has arbitrary small positive elements. Fix now $x \in R$. For any positive integer n there is $a \in Z$ such that $0 < a < 1/n$. We have $ma \leq x < (m+1)a$ for some integer m. Set $x_n = ma$. Then $|x - x_n| < 1/n$ and $x_n \in Z$. We see that x_n converges to x and $f(x_n) = 0$ for all n. By continuity of f we conclude that $f(x) = 0$. We see that if Z has no smallest positive element, then $f(x) = 0$ for all x.

We summarize the discussion so far as follows:

- Z5. if f is not identically 0 then $Z = \{na : n \in \mathbb{Z}\}$ for some $a > 0$.

From now on we assume that f is not identically 0. From Z5 and Z3 we see that f is periodic of period $2a$. Also, f has no zeros in the interval $(0,a)$, so f is either positive on $(0,a)$ or it is negative on $(0,a)$ (by continuity and the intermediate value theorem). Replacing f by $-f$ if necessary, we may and will assume that $f(x)>0$ for all $x\in(0,a)$. Since f is odd, f is negative on $(-a,0)$. Now if $a/2>x>y>0$ then $x+y\in(0,a)$ and $x-y\in(0,a)$. Thus $f^2(x)-f^2(y)=f(x+y)f(x-y)>0$, so $f(x)>f(y)$. In other words, f is increasing on $(0,a/2)$. Also, $f^2(x+a/2)-f^2(-x+a/2)=f(a)f(2x)=0$, so $f(a/2+x)=f(a/2-x)$ for $x\in(0,a/2)$. In particular, f is decreasing on $(a/2,a)$. Thus $f(a/2)$ is the largest value of f on $(0,a)$. Since f is odd and periodic of period $2a$, $f(a/2)$ is the largest value of f on $\mathbb{R}$. Thus $0\geq f^2(x+a/2)-f^2(a/2)=f(x+a)f(x)$. Since $f(x)=\pm f(x+a)$ by Z4, we conclude that $f(x+a)=-f(x)$ for all $x\in\mathbb{R}$. Thus

$$f(x+a/2)=-f(x+a/2-a)=-f(x-a/2)=f(a/2-x)$$

for all $x\in\mathbb{R}$. Let us summarize our observations.

Z6. The function f is odd and periodic of period $2a$, $f(x+a)=-f(x)$ and $f(a/2+x)=f(a/2-x)$ for all $x\in\mathbb{R}$. f is increasing on $(0,a/2)$ and decreasing of $(a/2,a)$ and $f(a/2)$ is the largest value of f.

Note now that $f^2(a/2)-f^2(x)=f(a/2+x)f(a/2-x)=f^2(a/2-x)$, so

$$f^2(x)+f^2(a/2-x)=f^2(a/2). \tag{3}$$

Also,

$$f^2(a/2-x)-f^2(x)=f(a/2)f(a/2-2x) \tag{4}$$

By now we should start to see that f resembles a familiar function, namely $\sin x$. It is not hard to see that $\sin x$ does indeed satisfy (1):

$$\sin^2 x-\sin^2 y=(\sin x-\sin y)(\sin x+\sin y)=2\sin((x-y)/2)\cos((x+y)/2)\cdot 2\sin((x+y)/2)\cos((x-y)/2)=$$

$$2\sin((x-y)/2)\cos((x-y)/2)\cdot 2\sin((x+y)/2)\cos((x+y)/2)=\sin(x-y)\sin(x+y).$$

Let $g(x)=f(a/2)\sin(\pi x/a)$. Then $g(x)$ satisfies (1), $g(a/2)=f(a/2)$, g and f have the same sets of zeros, and $g(x)$ and $f(x)$ have the same sign (i.e. are both positive, both negative, or both 0) for every $x\in\mathbb{R}$. We will show that $f=g$.

Let $T=\{a: f(a)=g(a)\}$. Thus $Z\subseteq T$ and $a/2\in T$. Since both f and g satisfy (3), and they have the same sign, we see that if $x\in T$ then $a/2-x\in T$. Suppose now that $2x\in T$. Thus $a/2-2x\in T$. From (3) and (4) we get $f^2(x)+f^2(a/2-x)=g^2(x)+g^2(a/2-x)$ and $f^2(a/2-x)-f^2(x)=g^2(a/2-x)-g^2(x)$, which implies that $f(x)=g(x)$ and $f(a/2-x)=g(a/2-x)$, i.e. both x and $a/2-x$ are in T. This means that if $x\in T$ then $x/2\in T$ and by obvious induction $x/2^m\in T$ for every positive integer m. In particular, for every integer n and positive integer m we have $na/2^m\in T$. Let now x be arbitrary. Given a positive integer m we can find an integer n such that $na/2^m\leq x<(n+1)a/2^m$. Set $x_m=na/2^m$. Then $|x-x_m|<a/2^m$. It follows that the sequence (x_m) converges to x and $x_m\in T$ for all m. Thus, since both f and g are continuous, we have

$$f(x)=\lim_{m\to\infty} f(x_m)=\lim_{m\to\infty} g(x_m)=g(x).$$

Since x was arbitrary, we proved that $f=g$ as promised.

In conclusion, f satisfies the conditions of the problem if and only if there are A and $B>0$ such that $f(x)=A\sin(Bx)$ for all $x\in\mathbb{R}$.

Remark. The assumption that f is continuous is necessary. For example, if $h(x)$ is a discontinuous additive function, then $f(x)=\sin h(x)$ is bounded and satisfies (1), but it is not continuous. For more about discontinuous additive functions see the solution to Problem 3 from Fall 2022.

Second solution. We assume that f is not identically 0. As in the first solution, we use the continuity and boundedness of f and (1) to observe that

$$Z = \{na : n \in \mathbb{Z}\} \text{ for some } a > 0,$$

where Z is the set of zeros of f. Moreover, replacing f by $-f$ if necessary, we may assume that f positive on $(0, a)$, increasing on $(0, a/2)$, and $f(a + x) = -f(x)$ for all x.

We will need the following fundamental theorem due to Lebesgue.

Lebesgue's Differentiation Theorem. *A function which is monotone on an open interval (a, b) is differentiable almost everywhere on (a, b).*

We will need a simpler version of this result which asserts that a monotone function on an open interval is differentiable at some point in the interval (though I am not aware of any proof of this seemingly simpler version which would not establish the whole Lebesgue's theorem).

By the Lebesgue's theorem, there is $u \in (0, a/2)$ at which f is differntiable. Note that $2u \in (0, a)$, so $f(2u) \neq 0$. By (1) we have

$$\frac{f(u+\epsilon) - f(u)}{\epsilon} \left(f(u+\epsilon) + f(u)\right) = \frac{f(\epsilon)}{\epsilon} f(2u + \epsilon).$$

Since $\lim_{\epsilon \to 0} \frac{f(u+\epsilon) - f(u)}{\epsilon} = f'(u)$ exists, $\lim_{\epsilon \to 0} f(u + \epsilon) = f(u)$, $\lim_{\epsilon \to 0} f(2u + \epsilon) = f(2u)$, and $f(2u) \neq 0$, we see that $\lim_{\epsilon \to 0} \frac{f(\epsilon)}{\epsilon}$ exists, it is equal to $f'(0)$ (since $f(0) = 0$), and

$$f'(u) \cdot 2f(u) = f(2u)f'(0).$$

Suppose now that $f(x) \neq 0$ (i.e. $x \notin Z$). Again by (1) we have

$$\frac{f(x+\epsilon) - f(x)}{\epsilon} \left(f(x+\epsilon) + f(x)\right) = \frac{f(\epsilon)}{\epsilon} f(2x + \epsilon).$$

Letting ϵ go to 0 and using the fact that $f(x) \neq 0$, we conclude that f is differentiable at x and

$$2f'(x)f(x) = f(2x)f'(0).$$

If $f(x) = 0$ then $x = na$ for some n. Since $f(x + na) = (-1)^n f(x)$ for all x, we have

$$\lim_{\epsilon \to 0} \frac{f(na+\epsilon) - f(na)}{\epsilon} = \lim_{\epsilon \to 0} \frac{(-1)^n f(\epsilon)}{\epsilon} = (-1)^n f'(0).$$

We see that f is differentiable at x as well.

To summarize the above discussion, we proved that f is differentiable at every x and

$$2f'(x)f(x) = f(2x)f'(0) \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

From (1) we get

$$f^2\left(\frac{x+y}{2}\right) - f^2\left(\frac{x-y}{2}\right) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. Differentiating this with respect to x we get

$$f\left(\frac{x+y}{2}\right) f'\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) f'\left(\frac{x-y}{2}\right) = f'(x)f(y)$$

for all $x, y \in \mathbb{R}$. Using (5), the last equality becomes

$$f'(0)\left(f(x+y) - f(x-y)\right) = 2f'(x)f(y) \tag{6}$$

for all $x, y \in \mathbb{R}$. Fixing y such that $f(y) \neq 0$, we conclude that f' is differentiable at all x. Differentiating (6) with respect to x we get

$$f'(0)\left(f'(x+y) - f'(x-y)\right) = 2f''(x)f(y) \tag{7}$$

for all $x, y \in \mathbb{R}$. Since f is odd, f' is even and therefore

$$2f''(y)f(x) = f'(0)\left(f'(y+x) - f'(y-x)\right) = f'(0)\left(f'(x+y) - f'(x-y)\right) = 2f''(x)f(y).$$

Thus we have $f''(x)f(y) = f(x)f''(y)$ for all $x, y \in \mathbb{R}$. If $x, y \notin Z$ then $f''(x)/f(x) = f''(y)/f(y)$. In other words, there is C such that $f''(x) = Cf(x)$ for all $x \notin Z$. If $x \in Z$ then $f(x) = 0$, so $f''(x)f(y) = 0$ for all y and therefore $f''(x) = 0$. We see that $f''(x) = Cf(x)$ holds for all x. Setting $x = a/2 = y$ in (7) we get

$$f'(0)(f'(a) - f'(0)) = 2f''(a/2)f(a/2).$$

Since $f'(a) = -f'(0)$, we get $-(f'(0))^2 = Cf(a/2)^2$. It follows that C is negative.

We showed that f satisfies the differential equation $f''(x) = Cf(x)$ for some $C < 0$. We may write $C = -b^2$ for some $b > 0$. Then the basic theory of ordinary differential equations tells us that $f(x) = A\sin(bx) + B\cos(bx)$ for some constants A, B. Since $f(0) = 0$ we conclude that $B = 0$ and $f(x) = A\sin(bx)$. Conversely, the function $f(x) = A\sin(bx)$ satisfies the conditions of the problem for any choice of A and b.

Remark. An elementary proof of Lebesgue's theorem can be found in the article *An Elementary Proof of Lebesgue's Differentiation Theorem* by Michael W. Botsko (The American Mathematical Monthly Vol. 110, No. 9 (Nov., 2003), pp. 834).

Problem. Use the ideas from our second solution to find all unbounded continuous functions which satisfy (1).
Challenge. Give a solution of the unbounded case without using Lebesgue's theorem.