Problem 5. Let (f_n) be the Fibonacci sequence: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for all $n > 2$. Prove that for every odd $n \geq 3$ the polynomial $x^n + f_n x^2 - f_{n-2}$ is divisible by $x^2 + x - 1$.

Solution 1. We are asked to show that

$$x^n + f_n x^2 - f_{n-2} = p_n(x)(x^2 + x - 1)$$

for some polynomial $p_n(x)$. Experimenting with several low values of n one can make a prediction that

$$p_n(x) = f_1 x^{n-2} - f_2 x^{n-3} + f_3 x^{n-4} - \cdots + f_{n-2} x + f_{n-2}$$

(recall that our assumption is that n is odd). Now performing the multiplication $p_n(x)(x^2 + x - 1)$ one verifies that this choice of the polynomial p_n works. A more systematic way to arrive at this solution is to consider a general polynomial $p_n(x) = c_1 x^{n-2} + c_2 x^{n-3} + \cdots + c_{n-1}$ of degree $n-2$ and observe that the product

$$p_n(x)(x^2 + x - 1) = d_1 x^n + d_2 x^{n-1} + \cdots + d_{n+1},$$

where $d_1 = c_1, d_2 = c_1 + c_2, d_k = c_k + c_{k-1} - c_{k-2}$ for $2 < k < n$, $d_n = c_{n-1} - c_{n-2}, d_{n+1} = -c_{n-1}$. Now setting $d_1 x^n + d_2 x^{n-1} + \cdots + d_{n+1} = x^n + f_n x^2 - f_{n-2}$ yields $d_1 = 1$, $d_k = 0$ for $1 < k < n-1$, $d_{n-1} = f_n$, $d_n = 0$, $d_{n+1} = -f_{n-2}$. Solving for c_i we see that $c_i = (-1)^{i-1} f_i$ for $1 \leq i < n-1$ and $c_{n-1} = f_{n-2}$.

Even better way to edit this solution would be to note that the polynomials p_n satisfy the following recursive formula: $p_3(x) = x + 1 = f_1 x + f_1$ and $p_{n+2}(x) = x^2 p_n(x) + f_n(-x^2 + x + 1)$. Then use induction to show that $x^n + f_n x^2 - f_{n-2} = p_n(x)(x^2 + x - 1)$ for every odd $n \geq 3$.

Remark. This solution is essentially the same as the solutions submitted by Sasha Aksenchuk, Mithun Padinhare Veettil, and Eric Wang.

Solution 2. Note that

$$x^2(x^n + f_n x^2 - f_{n-2}) = (x^{n+2} + f_{n+2} x^2 - f_n) + f_n x^4 - (f_{n+2} + f_{n-2}) x^2 + f_n.$$

Now note that $f_{n+2} + f_{n-2} = 3 f_n$. Indeed, $f_{n+2} = f_{n+1} + f_n$ and $f_{n-2} = f_n - f_{n-1}$ so

$$f_{n+2} + f_{n-2} = f_{n+1} + f_n + f_n - f_{n-1} = 2 f_n + f_{n+1} - f_{n-1} = 3 f_n.$$

Thus $f_n x^4 - (f_{n+2} + f_{n-2}) x^2 + f_n = f_n(x^4 - 3x^2 + 1) = f_n(x^2 + x - 1)(x^2 - x - 1)$. We see that

$$x^{n+2} + f_{n+2} x^2 - f_n = x^2(x^n + f_n x^2 - f_{n-2}) - f_n(x^2 + x - 1)(x^2 - x - 1).$$

Since $x^3 + f_3 x^2 - f_1 = (x^2 + x - 1)(x + 1)$, the result follows now by a starightforward induction.

Solution 3. Let $u = (1 + \sqrt{5})/2$ and $w = (1 - \sqrt{5})/2 = -1/u$, so u, w are the roots of $x^2 - x - 1 = 0$. We will use the following well known formula for the Fibonnaci numbers:

$$f_n = (u^n - w^n)/\sqrt{5}. \tag{1}$$

The roots of $x^2 + x - 1$ are $-u$ and $-w$. When n is odd we have

$$(-u)^n + f_n(-u)^2 - f_{n-2} = -u^n + \frac{u^n - w^n}{\sqrt{5}} u^2 - \frac{u^{n-2} - w^{n-2}}{\sqrt{5}} = -u^n + \frac{u^{n+2}}{\sqrt{5}} - \frac{u^{n-2}}{\sqrt{5}} = \frac{u^{n-2}}{\sqrt{5}} \left(u^4 - \sqrt{5} u^2 - 1 \right)$$

(we used the equality $w^2 u^2 = 1$). Since $u^2 - w^2 = \sqrt{5}$ (by (1) with $n = 2$), we see that

$$u^4 - \sqrt{5} u^2 - 1 = u^4 - (u^2 - w^2) u^2 - 1 = u^4 - u^4 + w^2 u^2 - 1 = 0.$$

This proves that $-u$ is a root of the polynomial $x^n + f_n x^2 - f_{n-2}$. Similarly we check that $-w$ is also a root of this polynomial. It follows that $x^n + f_n x^2 - f_{n-2}$ is divisible by $(x + u)(x + w) = x^2 + x - 1$.

Remark. This solution is essentially the same as the one submitted by Slava Kargin.

Excercise. Prove formula (1) by induction.

Solution 4. In this solution we will work with a more general situation. Consider a quadratic polynomial $x^2 + ax + b$. Let $n \geq 0$ be an integer. By the division algorithm for polynomials there are a unique polynomial $h_n(x)$ and unique numbers A_n, B_n such that

$$x^n = h_n(x)(x^2 + ax + b) + A_n x + B_n.$$

We have $h_0 = 0, A_0 = 0, B_0 = 1$, $h_1 = 0, A_1 = 1, B_1 = 0$. Now

$$x^{n+1} = x \cdot x^n = xh_n(x)(x^2 + ax + b) + A_n x^2 + B_n x =$$

$$xh_n(x)(x^2+ax+b)+A_n(x^2+ax+b)+(B_n-aA_n)x-bA_n = (xh_n(x)+A_n)(x^2+ax+b)+(B_n-aA_n)x-bA_n.$$

It follows that $h_{n+1}(x) = xh_n(x) + A_n$, $A_{n+1} = B_n - aA_n$, $B_{n+1} = -bA_n$. We see that

$$A_{n+2} = B_{n+1} - aA_{n+1} = -aA_{n+1} - bA_n,\ A_0 = 0,\ A_1 = 1,$$

$$B_n = -bA_{n-1} \text{ for } n \geq 1,$$

and

$$h_n(x) = A_1 x^{n-2} + A_2 x^{n-3} + \ldots + A_{n-2}x + A_{n-1} \text{ for } n \geq 2.$$

Let us now return to the problem, i.e. take $a = 1, b = -1$. We see that $A_{n+2} = -A_{n+1} + A_n$. In particular, $A_2 = -1$. This can be written as

$$(-1)^{n+1}A_{n+2} = (-1)^n A_{n+1} + (-1)^{n-1}A_n,\ (-1)^0 A_1 = 1, (-1)^1 A_2 = 1.$$

It is now clear that $(-1)^{n-1}A_n = f_n$, i.e. $A_n = (-1)^{n-1} f_n$ for all $n \geq 1$. Thus $B_n = A_{n-1} = (-1)^{n-2} f_{n-1}$. We see that for n odd we have

$$x^n = h_n(x)(x^2 + x - 1) + f_n x - f_{n-1} \tag{2}$$

so

$$h_n(x)(x_2+x-1) = x^n - f_n x + f_{n-1} = x^n - f_n(x^2+x-1) + f_n x^2 - f_n + f_{n-1} = -f_n(x^2+x-1) + x^n + f_n x^2 - f_{n-2},$$

i.e.

$$x^n + f_n x^2 - f_{n-2} = (h_n(x) + f_n)(x^2 + x - 1)$$

which proves the assertion of the problem.

Remark. If z is a root of $x^2 + x - 1$ then (2) yields $z^n = f_n z - f_{n-1}$ for odd n. The solution by Saurabh A. Chandorkar proves this by induction and then conludes from it that z is also a root of $x^n + f_n x^2 - f_{n-2}$, which is equivalent to our problem.

Excercise. Show that if $n \geq 4$ is even then $x^n - f_n x^2 + f_{n-2}$ is divisible by $x^4 - 3x^2 + 1$.