Problem 6. Find the limit

$$\lim_{n\to\infty} \sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{n+4+n^n}}}}}}.$$

Solution. Let

$$b_n = \sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{n+4+n^n}}}}}}.$$

Note that $n+4+n^n \geq n^2+2n+4 = (n+2)^2$ for $n \geq 3$. Define

$$a_n = \sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{(n+2)^2}}}}}}.$$

Then $b_n \geq a_n$ for all $n \geq 3$. Note that $a_2 = \sqrt{5+\sqrt{4^2}} = 3$, $a_3 = \sqrt{5+\sqrt{6+2\sqrt{5^2}}} = 3$. We claim that $a_n = 3$ for all $n \geq 2$. Indeed,

$$a_{n+1} = \sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{(n+4)+n\sqrt{(n+3)^2}}}}}}} =$$
$$\sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{(n+4)+n(n+3)}}}}}} =$$
$$\sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{(n+2)^2}}}}}} = a_n.$$

Thus $a_n = 3$ for all $n \geq 2$ by a straightforward induction. It follows that $b_n > 3$ for all $n \geq 3$.

On the other hand let $c_n = \dfrac{b_n}{n^{n/2^n}}$. Then

$$c_n = \sqrt{\frac{5}{n^{n/2^{n-1}}}+1\sqrt{\frac{6}{n^{n/2^{n-2}}}+2\sqrt{\frac{7}{n^{n/2^{n-3}}}+3\sqrt{\dots\sqrt{\frac{n+3}{n^{n/2}}+(n-1)\sqrt{\frac{n+4+n^n}{n^n}}}}}}}.$$

Since $n^{n/2^k} > 1$ for every positive integers n, k and $(n+4+n^n)/n^n < (n+2)^2$ for every positive integer n, we easily see that

$$c_n < \sqrt{5+1\sqrt{6+2\sqrt{7+3\sqrt{\dots\sqrt{(n+3)+(n-1)\sqrt{(n+2)^2}}}}}} = a_n = 3.$$

Thus $b_n < 3n^{n/2^n}$ for all $n \geq 2$.

To summarize, we showed that

$$3 < b_n < 3n^{n/2^n}$$

for every $n \geq 3$. Note that $\lim_{n\to\infty} n^{n/2^n} = 1$. By the squeeze theorem, we conclude that $\lim_{n\to\infty} b_n = 3$.

Remark. To see that $\lim_{n\to\infty} n^{n/2^n} = 1$ note that $n < 2^n$ for all n so

$$1 < n^{n/2^n} < 2^{n^2/2^n}$$

for all n. Since $\lim_{n\to\infty} n^2/2^n = 0$, we see that $\lim_{n\to\infty} 2^{n^2/2^n} = 1$ and therefore $\lim_{n\to\infty} n^{n/2^n} = 1$ by the squeeze theorem.