Problem 2. Find all positive integers n such that $9n+19$ and $3n^2+9n+5$ are both cubes of integers.

Solution. Suppose that $9n+19=a^3$ and $3n^2+9n+5=b^3$ for some positive integers a,b. Then

$$(ab)^3 = (9n+19)(3n^2+9n+5) = 27n^3+138n^2+216n+95.$$

Note that

$$(3n+4)^3 = 27n^3+108n^2+144n+64 < (ab)^3$$

and

$$(3n+6)^3 = 27n^3+162n^2+324n+216 > (ab)^3.$$

Thus we must have $(ab)^3=(3n+5)^3$, i.e.

$$27n^3+138n^2+216n+95 = 27n^3+135n^2+225n+125.$$

This is equivalent to $n^2-3n-10=0$, i.e. $n=5$. Now it is easy to verify that $n=5$ indeed is a solution: $9n+19=64=4^3$ and $3n^2+9n+5=125=5^3$.

Exercise. Is there a positive integer n such that $4n+1$ and $9n+1$ are squares?

Remark. It is true, but much harder to prove, that the only integers n such that $3n^2+9n+5$ is a cube are $n=-8,-2,-1,5$.