Problem 1. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $\lim\limits_{x\to\infty}(f(x) + 2f'(x)) = 1$. Prove that $\lim\limits_{x\to\infty} f'(x) = 0$.

Solution. The conclusion of the problem is equivalent to $\lim\limits_{x\to\infty} f(x) = 1$.

Let $h(x) = f(x)e^{x/2}$, $g(x) = e^{x/2}$. Then $h'(x) = (f'(x) + f(x)/2)e^{x/2}$ and $h'(x)/g'(x) = f(x) + 2f'(x)$. It follows that

$$\lim_{x\to\infty} \frac{h'(x)}{g'(x)} = \lim_{x\to\infty}(f(x) + 2f'(x)) = 1.$$

This suggests using L'Hospital's rule. Most calculus books state L'Hospital rule under the assumption that $\lim_{x\to\infty} g(x) = \pm\infty = \lim_{x\to\infty} h(x)$. However, the following version is true.

L'Hospital's Rule. *Suppose that h, g are differentiable functions on (a, ∞) such that $\lim_{x\to\infty} \frac{h'(x)}{g'(x)} = A$ (this, in particular, assumes that $g'(x) \neq 0$ for all sufficiently large x) and $\lim_{x\to\infty} g(x) = \pm\infty$. Then $\lim_{x\to\infty} \frac{h(x)}{g(x)} = A$.*

Our functions h, g satisfy the assumptions of the above theorem, so we get

$$1 = \lim_{x\to\infty} \frac{h(x)}{g(x)} = \lim_{x\to\infty} f(x),$$

as required.

For completeness, we provide the proof of the above version of L'Hospital's Rule after our second solution.

Second solution. Let $\epsilon > 0$. There is x_ϵ such that $|f(x) + 2f'(x) - 1| < \epsilon$ for all $x \geq x_\epsilon$. This means that

$$-\frac{1}{2}f(x) + \frac{1-\epsilon}{2} < f'(x) < -\frac{1}{2}f(x) + \frac{1+\epsilon}{2}$$

for all $x \geq x_\epsilon$.
Now consider function $h(x), g(x)$ which satisfy the differential equations

$$h'(x) = -\frac{1}{2}h(x) + \frac{1+\epsilon}{2}, \quad g'(x) = -\frac{1}{2}g(x) + \frac{1-\epsilon}{2}, \quad h(x_\epsilon) = f(x_\epsilon) = g(x_\epsilon).$$

It is easy to see that $h(x) = ce^{-x/2} + 1 + \epsilon$ and $g(x) = de^{-x/2} + 1 - \epsilon$ for appropriate constants c, d (obtained from the condition $h(x_\epsilon) = f(x_\epsilon) = g(x_\epsilon)$). Let $H(x) = h(x) - f(x)$ and $G(x) = f(x) - g(x)$. Then $H(x_\epsilon) = 0 = G(x_\epsilon)$, $H'(x) = h'(x) - f'(x) > -H(x)/2$, and similarly $G'(x) > -G(x)/2$. We claim that $H(x) \geq 0$ and $G(x) \geq 0$ for all $x > x_\epsilon$. Indeed, if $H(x) < 0$ for some x then consider $u \in [x_\epsilon, x]$ for which H assumes the smallest value. Then $u > x_\epsilon$ and $H(u) \leq H(x) < 0$. It follows that $H'(u) > 0$. This however means that $H(u - t) < H(u)$ for small $t > 0$, contradicting the choice of u. Similar argument shows $G(x) \geq 0$ for $x > x_\epsilon$. In other words, we proved that $h(x) \geq f(x) \geq g(x)$ for all $x > x_\epsilon$. Note that there is $w > x_\epsilon$ such that $-\epsilon < ce^{-x/2} < \epsilon$ and $-\epsilon < de^{-x/2} < \epsilon$ for $x > w$. This implies that $1 - 2\epsilon < f(x) < 1 + 2\epsilon$ for $x > w$. As ϵ is an arbitrary positive number, we conclude that $\lim_{x\to\infty} f(x) = 1$.

Proof of the above L'Hospitale's Rule. We start by recalling the following fundamental result.

Rolle's Theorem. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function which is differentiable on the open interval (a, b) and such that $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.*

Let us give the idea behind this theorem: f, being continuous on a closed interval $[a, b]$, attains its largest and smallest values and one of them must be attained at some $c \in (a, b)$. For such c we must have $f'(c) = 0$ (can you justify this?).

From Rolle's Theorem, we deduce the following very useful result.

Cauchy's Mean Value Theorem. *Let f, g be functions continuous on a closed interval $[a, b]$, differentiable on (a, b) and such that $g(a) \neq g(b)$ and whenever $g'(u) = 0$ then $f'(u) \neq 0$. Then there is $c \in (a, b)$ such that*

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let $h(x) = (f(a) - f(b))g(x) - (g(a) - g(b))f(x)$. Then $h(a) = h(b) = g(b)f(a) - g(a)f(b)$. Thus, by Rolle's theorem, there is $c \in (a, b)$ such that $h'(c) = 0$. This means that

$$(f(a) - f(b))g'(c) - (g(a) - g(b))f'(c) = 0.$$

If $g'(c) = 0$ then we would have $f'(c) = 0$ contrary to our assumption. It follows that $g'(c) \neq 0$ and therefore

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$

Now we can prove the L'Hospitale's Rule. Our assumptions imply that there is x_0 such that $g'(x) \neq 0$ for $x \geq x_0$. In particular, $g(x) \neq g(y)$ if $x \neq y$ and both x, y are $\geq x_0$. The proof is based on the following straightforward equality:

$$\frac{h(x) - h(u)}{g(x) - g(u)} - A = \left(\frac{h(x)}{g(x)} - A\right) \frac{g(x)}{g(x) - g(u)} + \frac{Ag(u) - h(u)}{g(x) - g(u)}. \tag{1}$$

Let $1 > \epsilon > 0$. Since $\lim_{x\to\infty} h'(x)/g'(x) = A$, there is $u = u_\epsilon > x_0$ such that $|h'(x)/g'(x) - A| < \epsilon/4$ for all $x > u$. The assumption that $\lim_{x\to\infty} g(x) = \pm\infty$ implies the existence of $w > u$ such that for all $x > w$ we have

$$\left|\frac{Ag(u) - h(u)}{g(x) - g(u)}\right| < \frac{\epsilon}{4} \quad \text{and} \quad \left|\frac{g(x)}{g(x) - g(u)}\right| \geq 1 - \frac{\epsilon}{2}.$$

Applying this to (1), we see that for $x > w$ we have

$$\left|\frac{h(x) - h(u)}{g(x) - g(u)} - A\right| \geq \left|\frac{h(x)}{g(x)} - A\right| \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4}.$$

On the other hand, by the Cauchy's Mean Value Theorem, we have

$$\left|\frac{h(x) - h(u)}{g(x) - g(u)} - A\right| = \left|\frac{h'(y)}{g'(y)} - A\right| < \frac{\epsilon}{4}$$

for some y between u and x. It follows that

$$\frac{\epsilon}{4} > \left|\frac{h(x)}{g(x)} - A\right| \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{4}$$

and therefore

$$\left|\frac{h(x)}{g(x)} - A\right| < \frac{\epsilon}{2 - \epsilon} < \epsilon$$

for all $x > w$ (recall that $\epsilon < 1$).
We showed that for any $\epsilon > 0$ there is w such that $|h(x)/g(x) - A| < \epsilon$ for all $x > w$. This means that $\lim_{x\to\infty} \frac{h(x)}{g(x)} = A$.

Excercise. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a twice differentiable function such that $\lim\limits_{x\to\infty} (f(x) + 4f'(x) + 4f''(x)) = 1$. Prove that $\lim\limits_{x\to\infty} f'(x) = 0$.

Exercise. Formulate and prove the analogous version of L'Hospitale's Rule for $\lim_{x\to a}$ instead of $\lim_{x\to\infty}$.