Problem 5. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R} \longrightarrow \mathbb{N}$ from the real numbers to natural numbers. Prove that there exists a sequence of functions $f_1, f_2, f_3, \ldots$ in $\mathcal{F}$ such that for every finite set $A \subseteq \mathbb{R}$ and every $g \in \mathcal{F}$ there is i such that $g(a) = f_i(a)$ for every $a \in A$.

Solution. While it does not matter much, let us start by remarking that we include 0 in the set of natural numbers. There has been long ongoing discussion weather 0 should be considered a natural number, and the opinions are still divided. For us $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers and $\mathbb{N}^+ = \{1, 2, \ldots\}$ is the set of positive integers.

Let $p_1, p_2, p_3, \ldots$ be the sequence of all prime numbers larger than 3, so $p_1 = 5, p_2 = 7, p_3 = 11$, etc. Any positive integer n can be written as $n = 2^N 3^M p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$, where $N, M, k_1, \ldots, k_s$ are natural numbers. This expression is essentially unique, which means that the non-zero exponents are uniquely determined by n (we can always add factors of the form p_m^0). Define the function $f_n : \mathbb{R} \longrightarrow \mathbb{N}$ as follows:

$$f_n(t) = \begin{cases} 0, & \text{if } t < \frac{-M-1+i}{N+1} \\ k_i, & \text{if } \frac{-M-1+i}{N+1} \leq t < \frac{-M+i}{N+1} \\ 0, & \text{if } \frac{-M+s}{N+1} \leq t. \end{cases}$$

For example, for $n = 3300 = 2^2 \cdot 3^1 \cdot 5^2 \cdot 7^0 \cdot 11^1$ we have $N = 2$, $M = 1$ and

$$f_{3300}(t) = \begin{cases} 0, & \text{if } t < \frac{-1}{3} \\ 2, & \text{if } \frac{-1}{3} \leq t < \frac{0}{3} = 0 \\ 0, & \text{if } 0 = \frac{0}{3} \leq t < \frac{1}{3} \\ 1, & \text{if } \frac{1}{3} \leq t < \frac{2}{3} \\ 0, & \text{if } \frac{2}{3} \leq t. \end{cases}$$

We claim that the sequence $f_1, f_2, \ldots$ has the required property. Indeed, consider real numbers $a_1 < a_2 < \ldots < a_k$ and a function $g \in \mathcal{F}$. There is a natural number N such that $1/(N+1) < a_{i+1} - a_i$ for $i = 1, \ldots, k-1$. Let m_i be the largest integer such that $\dfrac{m_i}{N+1} \leq a_i$. Then $a_i < \dfrac{m_i+1}{N+1} < a_{i+1}$. Thus $m_1 < m_2 < \ldots < m_k$. Set $M = \max\{-m_1, 0\}$ and $s = M+1+m_k$. Define natural numbers $k_1, k_2, \ldots, k_s$ as follows. If $j = M+1+m_i$ set $k_j = g(a_i)$. For all other j set $k_j = 0$. Take $n = 2^N 3^M p_1^{k_1} \ldots p_s^{k_s}$. With this notation, we have

$$\frac{-M-1+(M+1+m_i)}{N+1} \leq a_i < \frac{-M+(M+1+m_i)}{N+1}$$

so $f_n(a_i) = k_{M+1+m_i} = g(a_i)$.

Second solution. The above solution uses a bit of trickery to get explicit formula for f_n. The solutions submitted by the solvers and our second solution produce a countable family of functions which satisfy the required property without any explicit order to list them as a sequence. The details of each of these solutions are different, but they all produce countably many functions which are step functions with finitely many steps.

The solutions are based on the following well known facts:

1. the set of rational numbers is countable
2. a subset of a countable set is countable
3. the union of countably many countable sets is countable
4. the set of all finite sequences with values in a countable set is countable.

Consider the set Λ of all sequences of the form $q_1, \ldots, q_m, k_1, \ldots, k_{m-1}$, where $m \geq 2$ is a natural number, $q_1 < q_2 < \ldots < q_m$ are rational numbers is ascending order and $k_1, \ldots, k_{m-1}$ are natural numbers. The set Λ is countable by 1-4 above. For any such sequence $\alpha \in \Lambda$ define a function f_α as follows

$$f_\alpha(t) = \begin{cases} 0, & \text{if } t < q_1 \\ k_i, & \text{if } q_i \leq t < q_{i+1} \\ 0, & \text{if } q_i \leq t. \end{cases}$$

Given real numbers $a_1 < a_2 < \ldots < a_m$ and a function $g \in \mathcal{F}$ there are rational numbers $q_1, \ldots, q_{m+1}$ such that $q_1 < a_1 < q_2 < a_2 < \ldots < q_m < a_m < q_{m+1}$. Set $k_i = g(a_i)$, $i = 1, \ldots, m$. For the sequence $\alpha = q_1, \ldots, q_{m+1}, k_1, \ldots, k_m$ we have $f_\alpha(a_i) = k_i = g(a_i)$. Thus the countable set $\{f_\alpha : \alpha \in \Lambda\}$ has the required property.

Remark. For those with some basic knowledge of topology, the above problem has an application in topology. The set $\mathcal{F}$ can be identified with the product $\mathbb{N}^{\mathbb{R}}$ (of continuum many copies of $\mathbb{N}$). If we consider $\mathbb{N}$ as topological space with the discrete topology, $\mathcal{F}$ becomes a topological space with the product topology. A sequence of functions $f_1, f_2, \ldots$ has the required property if and only if the set $\{f_1, f_2, \ldots\}$ is dense in $\mathcal{F}$. Recall that a topological space with a countable dense subset is called separable. Thus our problem is equivalent to the following result: the space $\mathbb{N}^{\mathbb{R}}$ is separable.

Problem. Use the above discussion to prove the following result: if X_i is a separable topological space for $i \in I$ and the cardinality $|I|$ of the set I does not exceed $|\mathbb{R}| = \mathfrak{c}$ then the product $\prod_{i \in I} X_i$ is separable (in the product topology). **Hint.** For each $i \in I$ consider a continuous function $h_i : \mathbb{N} \longrightarrow X_i$ whose image is dense in X_i.

Problem. Prove that if $|I| > \mathfrak{c}$ then the result of the previous problem is no longer true in general.