

**Problem 7.** Prove that the number

$$\left(\sqrt{13 + \sqrt{52}} - \sqrt{13}\right) (\sqrt{14 + \sqrt{52}} - \sqrt{13}) (\sqrt{15 + \sqrt{52}} - \sqrt{13})$$

is irrational.

**Solution.** This problem is more form than substance. The key observation is that

$$14 + \sqrt{52} = 13 + 2\sqrt{13} + 1 = (\sqrt{13} + 1)^2$$

and therefore  $\sqrt{14 + \sqrt{52}} - \sqrt{13} = 1$ . It follows that the number in question is equal to  $\sqrt{13 + \sqrt{52}} - \sqrt{13}$ . In order to prove that this number is irrational, assume for the sake of contradiction that

$$\sqrt{13 + \sqrt{52}} - \sqrt{13} = a$$

is rational. Then  $\sqrt{13 + 2\sqrt{13}} = a + \sqrt{13}$  and consequently  $13 + 2\sqrt{13} = a^2 + 2a\sqrt{13} + 13$ . Thus  $2(1 - a)\sqrt{13} = a^2$ . Note that  $a$  can not be 1 (otherwise we would have  $0 = 1$ ), so

$$\sqrt{13} = \frac{a^2}{2(1 - a)}.$$

Since  $a$  is a rational number, we conclude that  $\sqrt{13}$  is a rational number, a contradiction.

**Exercise.** Prove that  $\sqrt{13}$  is not rational.