Problem 5. The sequence (a_n) is defined recursively as follows: $a_1 = 1$, $a_{n+1} = \sin a_n$. Prove that the sequence $(\sqrt{n}a_n)$ converges and find its limit.

Solution. Proving convergence is often related to estimates. Since our sequence is build based on the sin function, recall that

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots.$$

Since the coefficients of the power series alternate in sign and decrease in absolute value, one can deduce that

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} \tag{1}$$

for all $x \in (0, 1)$. As a matter of fact, (1) holds for all $x > 0$. A simple proof of these inequalities will be given at the end of our solution. Recall also that $\sin x$ is increasing on $(-\pi/2, \pi/2)$.

If the sequence $(\sqrt{n}a_n)$ converges, then it must be bounded below. Let k be a non-negative integer and $c > 0$. Let us try to prove that $a_n > c/\sqrt{n+k}$ for every n using induction. We would need $a_1 = 1 > c/\sqrt{k+1}$, which can be achieved by choosing k large enough. Now we would like to make the inductive step: assuming that $a_n > c/\sqrt{n+k}$ we would like to argue that $a_{n+1} > c/\sqrt{n+1+k}$. Recall that $a_{n+1} = \sin a_n$ and $\sin x$ is increasing on $(0, 1)$. Thus, by (1), we have

$$a_{n+1} = \sin a_n > \sin\left(\frac{c}{\sqrt{n+k}}\right) > \frac{c}{\sqrt{n+k}}\left(1 - \frac{c^2}{6(n+k)}\right).$$

It would suffice to verify that

$$\frac{c}{\sqrt{n+k}}\left(1 - \frac{c^2}{6(n+k)}\right) > \frac{c}{\sqrt{n+1+k}}$$

which is equivalent to

$$\left(1 - \frac{c^2}{6(n+k)}\right)^2 > 1 - \frac{1}{n+k+1}$$

i.e.

$$\frac{c^2}{3(n+k)} - \frac{c^4}{36(n+k)^2} < \frac{1}{n+k+1}.$$

It is easy to see that if $c^2/3 \geq 1$, i.e. $c \geq \sqrt{3}$, then the last inequality can not hold for all n (multiply both sides by n and let n tend to infinity). However, if $c^2 < 3$ and $c^2/(3(1+k)) < 1/(k+2)$, i.e. $c^2 < 3(k+1)/(k+2)$, which can be achieved by taking k sufficiently large, then we have

$$\frac{c^2}{3(n+k)} < \frac{1}{n+k+1}$$

for every $n \geq 1$. This means that the induction works in this case. In other words, we showed the following

If $0 < c < \sqrt{3}$ and k is an integer such that $c^2 < 3(k+1)/(k+2)$ and $\sqrt{1+k} > c$ (any $k > c^2/(3-c^2)$ has this property) then

$$\sqrt{n+k}a_n > c \tag{2}$$

for all $n \geq 1$.

This in particular implies that if the limit $\lim_{n\to\infty}\sqrt{n}a_n$ exists then it must be at least $\sqrt{3}$, as for any $c < \sqrt{3}$ we have

$$\lim_{n\to\infty}\sqrt{n}a_n = \lim_{n\to\infty}\sqrt{n+k}a_n\frac{\sqrt{n}}{\sqrt{n+k}} \geq c\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+k}} = c$$

For $c = \sqrt{3}$ the argument above will not work. Perhaps then we can try to prove that $a_n \leq \sqrt{3}/\sqrt{n+k}$. To have this for $n = 1$ we need to take $k \leq 2$. Let us then take $k = 2$ and let us try to make the inductive step: assuming that $a_n \le \sqrt{3}/\sqrt{n+2}$ we would like to argue that $a_{n+1} \le \sqrt{3}/\sqrt{n+3}$. Again, $a_{n+1} = \sin a_n$ and $\sin x$ is increasing on $(0,1)$. Thus, by (1), we have

$$a_{n+1} = \sin a_n \le \sin\left(\frac{\sqrt{3}}{\sqrt{n+2}}\right) < \frac{\sqrt{3}}{\sqrt{n+2}}\left(1 - \frac{3}{6(n+2)} + \frac{9}{120(n+2)^2}\right) = \frac{\sqrt{3}}{\sqrt{n+2}}\left(1 - \frac{1}{2(n+2)} + \frac{3}{40(n+2)^2}\right).$$

It would suffice to verify that

$$\frac{\sqrt{3}}{\sqrt{n+2}}\left(1 - \frac{1}{2(n+2)} + \frac{3}{40(n+2)^2}\right) \le \frac{\sqrt{3}}{\sqrt{n+3}}$$

which, after substitution $t = n+2$, is equivalent to

$$\left(1 - \frac{1}{2t} + \frac{3}{40t^2}\right)^2 \le 1 - \frac{1}{t+1}.$$

Now

$$\left(1 - \frac{1}{2t} + \frac{3}{40t^2}\right)^2 = 1 - \frac{1}{t} + \frac{2}{5t^2} - \frac{3}{40t^3}\left(1 - \frac{3}{40t}\right) \le 1 - \frac{1}{t} + \frac{2}{5t^2}$$

when $t > 1$. Thus it would suffice to show that

$$1 - \frac{1}{t} + \frac{2}{5t^2} \le 1 - \frac{1}{t+1}$$

i.e. that

$$\frac{2}{5t^2} \le \frac{1}{t} - \frac{1}{t+1} = \frac{1}{t(t+1)}.$$

Since $2/5t < 1/(t+1)$ for any $t > 1$, we see that the inductive step indeed works. Thus we proved that

$$a_n \le \frac{\sqrt{3}}{\sqrt{n+2}} \tag{3}$$

for every $n \ge 1$.

It is not hard, but requires some work, to see that (2) and (3) together imply that $\sqrt{n}a_n$ converges to $\sqrt{3}$. We will use a slightly different argument which gives us additional information about the sequence $\sqrt{n}a_n$. Note that (3) implies that $\sqrt{n}a_n < \sqrt{3}$ for all n and therefore if $\sqrt{n}a_n$ converges then its limit does not exceed $\sqrt{3}$. Before we proved that if $\sqrt{n}a_n$ converges then its limit is at least $\sqrt{3}$. In other words, if $\sqrt{n}a_n$ converges then its limit is $\sqrt{3}$.

It remains to prove that $\sqrt{n}a_n$ converges. It suffices to prove that $\sqrt{n}a_n$ is increasing (recall that every increasing and bounded above sequence converges). We will prove that $\sqrt{n+1}a_{n+1} = \sqrt{n+1}\sin a_n > \sqrt{n}a_n$. By (1), we have $\sin a_n > a_n(1 - a_n^2/6)$, so it suffices to show that

$$\sqrt{n+1}a_n(1 - a_n^2/6) > \sqrt{n}a_n$$

i.e.

$$\left(1 - \frac{a_n^2}{6}\right)^2 = 1 - \frac{a_n^2}{3} + \frac{a_n^4}{36} > 1 - \frac{1}{n+1}$$

which would follow if $a_n^2 < 3/(n+1)$. The last inequality follows from (3), which completes our proof that $\sqrt{n}a_n$ is increasing.

Using the ideas from our first solution one can get a better estimate for a_n:

Exercise. Prove that

$$\frac{\sqrt{3}}{\sqrt{n+8+\ln n}} < a_n < \frac{\sqrt{3}}{\sqrt{n+\ln\ln n}}$$

for all $n \ge 2$

Second solution. Our second solution will be much shorter, based on the following important result due to Cauchy (1821):

Theorem (Cauchy). *Suppose that* $lim_{n\to\infty} s_n = L$*. Define a new sequence* (t_n) *by*

$$t_n = \frac{s_1 + \ldots + s_n}{n}.$$

Then $lim_{n\to\infty} t_n = L$*.*

A proof of this theorem will be given at the end for completeness.

It is easy to see that Cauchy's theorem is equivalent to the following result:

$$if \lim_{n\to\infty}(g_{n+1} - g_n) = L \; then \; \lim_{n\to\infty} g_n/n = L. \tag{4}$$

Indeed, take $s_1 = g_1$ and $s_n = g_n - g_{n-1}$ for $n > 1$. Then $g_n = s_1 + \ldots + s_n$.

We apply (4) to the sequence $g_n = 1/a_n^2$. We have

$$\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} = \frac{a_n^2 - a_{n+1}^2}{a_n^4}\frac{a_n^2}{a_{n+1}^2} = \frac{a_n - \sin a_n}{a_n^3}\frac{a_n + \sin a_n}{a_n}\left(\frac{a_n}{\sin a_n}\right)^2.$$

Note that $\lim_{x\to 0} x/\sin x = 1$. Using either l'Hopital's rule or (1) we see that $\lim_{x\to 0}(x - \sin x)/x^3 = 1/6$. Since a_n converges to 0 (see solution to Problem 4), we conclude that

$$\lim_{n\to\infty}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right) = \lim_{n\to\infty}\left(\frac{a_n - \sin a_n}{a_n^3}\right)\lim_{n\to\infty}\left(\frac{a_n + \sin a_n}{a_n}\right)\left(\lim_{n\to\infty}\frac{a_n}{\sin a_n}\right)^2 = 1/3.$$

It follows that $\lim_{n\to\infty} 1/na_n^2 = 1/3$, hence $\lim_{n\to\infty}\sqrt{n}a_n = \sqrt{3}$.

Exercise. Let f be a function with continuous derivatives up to order $k \geq 2$ such that $f(0) = 0$, $f'(0) = 1$, $f^{(j)}(0) = 0$ for $j = 2, \ldots, k-1$ and $f^{(k)}(0) = A \neq 0$. Suppose a is such that the sequence (a_n) defined recursively by $a_1 = a$, $a_{n+1} = f(a_n)$ converges to 0 and $a_n \neq 0$ for all n. Prove that

$$\lim_{n\to\infty} \sqrt[k-1]{n}a_n = \sqrt[k-1]{\frac{(k-1)!}{-A}}$$

Exercise. The reader may remember from calculus that "if the ratio test works then the root test works". In other words, if (a_n) is a sequence with positive terms such that $\lim_{n\to\infty} a_{n+1}/a_n = L$ then $\lim_{n\to\infty}\sqrt[n]{a_n} = L$. Prove that this fact is equivalent to (4).

Proof of (1). Let $f(x) = x - x^3/6 + x^5/120 - \sin x$. Then $f'(x) = 1 - x^2/2 + x^4/24 - \cos x$, $f''(x) = -x + x^3/6 + \sin x$, $f^{(3)}(x) = -1 + x^2/2 + \cos x$, $f^{(4)}(x) = x - \sin x$, $f^{(5)}(x) = 1 - \cos x$. Since $f^{(5)}(x) \geq 0$ we see that $f^{(4)}(x)$ is increasing. Since $f^{(4)}(0) = 0$, we conclude that $f^{(4)}(x) > 0$ for $x > 0$. Thus $f^{(3)}(x)$ is increasing on $(0, \infty)$. Since $f^{(3)}(0) = 0$, we conclude that $f^{(3)}(x) > 0$ on $(0, \infty)$. This tells us that $f''(x)$ is increasing on $(0, \infty)$. Since $f''(0) = 0$, we conclude that $f''(x) > 0$ on $(0, \infty)$. This proves the left inequality of (1). Furthermore, $f'(x)$ is is increasing on $(0, \infty)$. Since $f'(0) = 0$, we conclude that $f'(x) > 0$ on $(0, \infty)$. Thus $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0$, we conclude that $f(x) > 0$ on $(0, \infty)$. This proves the right inequality of (1).

Proof of Cauchy's Theorem. Fix $\epsilon > 0$. Since $\lim_{n\to\infty} s_n = L$, there is positive integer k such that $|s_n - L| < \epsilon/2$ for all $n \geq k$. Thus, for $n \geq k$, we have

$$|t_n - L| = \left|\frac{s_1 + \ldots + s_n}{n} - L\right| = \left|\frac{(s_1 - L) + \ldots + (s_n - L)}{n}\right| \leq \left|\frac{(s_1 - L) + \ldots + (s_k - L)}{n}\right| + \frac{n-k}{n}\frac{\epsilon}{2}.$$

Take N large enough so that

$$\left|\frac{(s_1 - L) + \ldots + (s_k - L)}{N}\right| < \frac{\epsilon}{2}.$$

Then for all $n \geq N$ we have

$$|t_n - L| < \epsilon.$$

This proves that $\lim_{n\to\infty} t_n = L$

Remark. Cauchy's Theorem is a basis for the so called Cesaro summation. A vast generalization of this theorem is the following result due to Toeplitz.

Theorem (Toeplitz). *Suppose that for every n we have given numbers $c_{n,1}, c_{n,2}, \ldots, c_{n,n}$ which satisfy the following conditions:*

1. $\lim_{n\to\infty} c_{n,k} = 0$ *for every k.*
2. *there is $B > 0$ such that $|c_{n,1}| + |c_{n,2}| + \ldots + |c_{n,n}| < B$ for all n.*
3. $\lim_{n\to\infty}(c_{n,1} + c_{n,2} + \ldots + c_{n,n}) = 1$

For any sequence (s_n) define a new sequence (t_n) by

$$t_n = c_{n,1}s_1 + c_{n,2}s_2 + \ldots + c_{n,n}s_n.$$

If $\lim_{n\to\infty} s_n = L$ then $\lim_{n\to\infty} t_n = L$.

Taking $c_{n,k} = 1/n$ we seee that Cauchy's theorem is a special case of Teoplitz's theorem.

Exercise. Use Toeplitz's theorem to prove the following result known as Stolz Theorem (or sometimes Stolz-Cesaro Theorem):

Stolz Theorem. *Suppose (g_n) and (h_n) are two sequences of real numbers such that (h_n) is strictly monotone (starting from some point on) and $\lim\limits_{n\to\infty} \frac{g_{n+1} - g_n}{h_{n+1} - h_n} = L$. If (h_n) is unbounded then $\lim\limits_{n\to\infty} \frac{g_n}{h_n} = L$.*

Stolz Theorem is also true when the condition that (h_n) is unbounded is replaced by the condition that both (g_n) and (h_n) converge to 0. It can be viewed as an analog for sequences of the l'Hopital's rule.