Problem 4. a) Let f be a differentiable function such that $f(\sin x) = \sin f(x)$ for every $x \in \mathbb{R}$. Prove that if f is not identically zero then $\lim_{x\to 0} \frac{f(x)}{x}$ exists and is equal to 1 or -1.

b) Prove that there is a continuous function f such that $f(\sin x) = \sin f(x)$ and $\lim_{x\to 0^+} \frac{f(x)}{x}$ does not exist.

Solution. Let us first make some simple observations about the function $\sin x$. Recall that the only solution to $\sin x = x$ is $x = 0$. Moreover $\sin x < x$ for every $x > 0$ and $\sin x > x$ for every $x < 0$. Also, $\sin x$ is increasing on $(-\pi/2, \pi/2)$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

For any number a define a sequence (a_n) as follows: $a_0 = a$ and $a_{n+1} = \sin a_n$. It is straightforward to see that if $a \in (0, \pi)$ then $a_n > 0$ for all n and therefore $0 < a_{n+1} < a_n$ for all n. A decreasing and bounded below sequence is convergent, so $\lim_{n\to\infty} a_n = g$ exists. Thus $\lim_{n\to\infty} \sin a_n = \sin g$. On the other hand, $\sin a_n = a_{n+1}$, so $\lim_{n\to\infty} \sin a_n = g$. We conclude that $\sin g = g$, i.e. $g = 0$. To summarize, if $a \in (0, \pi)$ then the sequence a_n is decreasing and converges to 0. Similarly, if $a \in (-\pi, 0)$, the sequence a_n is increasing and converges to 0 (note that if $b = -a$ then $b_n = -a_n$ for all n.)

Taking $a = 1$, we get the sequence 1_n. If $0 < a < 1$ then a straightforward induction yields $a_n < 1_n$ for all n (as $\sin x$ is increasing on $(0, 1)$). On the other hand, there is k such that $1_k < a = a_0$ (why?). Again a straightforward induction yields $1_{k+n} < a_n$ for all n. Thus

$$\frac{1_{k+n}}{1_n} < \frac{a_n}{1_n} < \frac{1_n}{1_n} = 1$$

for all n. If we prove that $\lim_{n\to\infty} \frac{1_{k+n}}{1_n} = 1$, then by pinching theorem we conclude that $\lim_{n\to\infty} \frac{a_n}{1_n} = 1$. Note that $\frac{1_{k+n}}{1_n} = \frac{1_{k+n}}{1_{k+n-1}} \frac{1_{k+n-1}}{1_{k+n-2}} \cdots \frac{1_{n+1}}{1_n}$. It suffices to show $\lim_{n\to\infty} \frac{1_{1+n}}{1_n} = 1$. This is clear, since 1_n tends to 0 and $1_{n+1} = \sin 1_n$ (recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$). Thus, we proved that for any $0 < a \le 1$ we have $\lim_{n\to\infty} \frac{a_n}{1_n} = 1$. In the same way, we show that for any $0 > a \ge -1$ we have $\lim_{n\to\infty} \frac{a_n}{1_n} = -1$.

First solution to a) Let $f(x)$ be a differentiable at 0 function such that $f(\sin x) = \sin f(x)$. As $f(0) = f(\sin 0) = \sin f(0)$, we have $f(0) = 0$. It follows that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

In other words, the limit on the right exists and it is equal to $f'(0)$.

If $x \in [-1, 1]$ then $x = \sin u$ for some u and $f(x) = f(\sin u) = \sin f(u) \in [-1, 1]$. Thus f maps $[-1, 1]$ to itself.

Suppose now that $f(a) \neq 0$ for some $a \in [-1, 1]$. Note that $f(a_n) = f(a)_n$ for all n. Thus

$$f'(0) = \lim_{n\to\infty} \frac{f(a_n)}{a_n} = \lim_{n\to\infty} \frac{f(a)_n}{1_n} \frac{1_n}{a_n} = \begin{cases} 1 & \text{if } a \text{ and } f(a) \text{ have the same sign} \\ -1 & \text{if } a \text{ and } f(a) \text{ have opposite signs.} \end{cases}$$

Suppose that $f(a) = 0$ for all $a \in [-1, 1]$. For any x we have $\sin x \in [-1, 1]$, so $\sin f(x) = f(\sin x) = 0$. Thus $f(x) \in \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$ for all x. As f is continuous, it has to be constant and therefore identically 0 (since $f(0) = 0$). This proves part a).

Second solution to a), based on ideas from solution by Ashton Keith. This solution works only when $f'(x)$ is continuous at 0. As in the first solution, we observe that the limit exists and is equal to $f'(0)$. Suppose that $f'(0)$ is different from $0, -1, 1$.
Taking derivatives of both sides of $f(\sin x) = \sin f(x)$ yields

$$\cos x f'(\sin x) = f'(x) \sin f(x).$$

Assuming $\cos x \neq 0$, we can rewrite it as

$$f'(x) - f'(\sin x) = f'(x)\frac{\cos x - \cos f(x)}{\cos x} = -2\sin\left(\frac{x - f(x)}{2}\right)\sin\left(\frac{x + f(x)}{2}\right)\frac{f'(x)}{\cos x}.$$

Since $\lim_{x\to 0}\frac{f(x)}{x} = f'(0) \neq \pm 1$, we get

$$\lim_{x\to 0}\frac{1}{x}\sin\left(\frac{x - f(x)}{2}\right) = \frac{1 - f'(0)}{2} \quad \text{and} \quad \lim_{x\to 0}\frac{1}{x}\sin\left(\frac{x + f(x)}{2}\right) = \frac{1 + f'(0)}{2}.$$

It follows that

$$\lim_{x\to 0}\frac{f'(x) - f'(\sin x)}{x^2} = \frac{f'(0)(f'(0)^2 - 1)}{2} = 2D \neq 0$$

(we used here the continuity of f' at 0 for the first time). We conclude that there is $\epsilon > 0$ such that

$$f'(x) - f'(\sin x) > Dx^2 \quad \text{for all } x \in (-\epsilon, \epsilon).$$

if $D > 0$ and

$$f'(x) - f'(\sin x) < Dx^2 \quad \text{for all } x \in (-\epsilon, \epsilon).$$

if $D < 0$. We will focus on the case $D > 0$ (the case $D < 0$ is handled same way). Take any $x < 1$ such that $0 < x < \epsilon$. Consider the sequence x_n (recall that $x_0 = x$ and $x_{n+1} = \sin x_n$). We know that this sequence is decreasing and tends to 0. There is n such that $2x_{n+1} < x$. Note that

$$f'(x) - f'(x_{n+1}) = \sum_{k=0}^{n}(f'(x_k) - f'(x_{k+1})) = \sum_{k=0}^{n}(f'(x_k) - f'(\sin x_k)) > D\sum_{k=0}^{n}x_k^2.$$

It is not hard to see that $t - \sin t \leq t^3/6$ for all $t > 0$. Thus

$$\frac{x}{2} \leq x_0 - x_{n+1} = \sum_{k=0}^{n}(x_k - x_{k+1}) = \sum_{k=0}^{n}(x_k - \sin x_k) \leq \frac{1}{6}\sum_{k=0}^{n}x_k^3 \leq \frac{x}{6}\sum_{k=0}^{n}x_k^2$$

(we used the straightforward inequality $x_k^3 < xx_k^2$). It follows that $\sum_{k=0}^{n}x_k^2 \geq 3$ and therefore

$$f'(x) - f'(x_{n+1}) \geq 3D.$$

On the other hand, we have $0 < x_{n+1} < x$, so letting x approach 0 and using the continuity of f' at 0 we get that $\lim_{x\to 0}(f'(x) - f'(x_{n+1})) = 0$. Thus $0 \geq 3D$, a contradiction.

The contradiction was obtained from the assumption that $f'(0)$ is different from $-1, 0, 1$. In order to complete the solution we need to exclude the case $f'(0) = 0$. If $f'(0) = 0$ then $|f(x)| < |x|$ for all x which are sufficiently small. It follows that $\cos f(x) > \cos x > 0$ and therefore $|f'(\sin x)| \geq |f'(x)|$ (recall that $\cos x f'(\sin x) = f'(x)\sin f(x)$) for all x small enough. This easily implies that $|f'(x_n)| \geq |f'(x)|$ for all n and all x small enough. Since x_n tends to 0, $f'(0) = 0$, and f' is continuous at 0, we conclude that $f'(x) = 0$ for all x sufficiently small. This implies that f is constant and therefore identically 0 on a small neighborhood of 0. Using the equality $f(\sin x) = \sin f(x)$ one can easily conclude that $f(x) = 0$ for all $x \in [-1, 1]$ (see the solution to part b) for a detailed idea how to do that) and then, as in our first solution, that $f(x) = 0$ for all x.

Remark. From Problem A3 from the 2020 Putnam Exam one can easily see that the series $\sum_{k=0}^{\infty}x_k^2$ diverges.

Solution to part b). Let $I = [-\pi/2, -1) \cup (1, \pi/2]$ and let $h : I \longrightarrow \mathbb{R}$ be any function. Recall the sequence 1_n: $1_0 = 1$ and $1_{n+1} = \sin 1_n$. This sequence decreases to 0. Define $I_0 = I$ and $I_k = [-1_{k-1}, -1_k) \cup (1_k, 1_{k-1}]$ for $k = 1, 2, ...$ The sets I_k are pairwise disjoint and their union is $[-\pi/2, 0) \cup (0, \pi/2]$. Not that for $k > 0$ and $u \in I_k$ we have $\arcsin u \in I_{k-1}$. Let $h_0 = h$ and define functions $h_k : I_k \longrightarrow R$ inductively by $h_{k+1}(u) = h_k(\arcsin u)$. Finally define $f : [-\pi/2, \pi/2] \longrightarrow \mathbb{R}$ by $f(0) = 0$ and $f(x) = h_k(x)$ if $x \in I_k$. Note that if $x \in I_k$ then $\sin x \in I_{k+1}$ and

$$f(\sin x) = h_{k+1}(\sin x) = h_k(\arcsin(\sin x)) = h_k(x) = f(x).$$

We see that f is the unique function on $[-\pi/2, \pi/2]$ which agrees with h on I and satisfies $f(\sin x) = \sin f(x)$ for all $x \in [-\pi/2, \pi/2]$. Note that for f to be continuous, a necessary condition is that h is continuous and $\sin h(\pi/2) = \lim_{x\to 1^-} h(x)$ and $\sin h(-\pi/2) = \lim_{x\to -1^+} h(x)$. In other words, h extends to a continuous function on $[-\pi/2, -1] \cup [1, \pi/2]$ and $h(\pm 1) = \sin h(\pm\pi/2)$. It is easy to see that this condition is also sufficient, i.e. the corresponding function f is continuous on $[-\pi/2, \pi/2]$. Thus we have the following key observation:

Proposition. For any continuous function $h : [-\pi/2, -1] \cup [1, \pi/2] \longrightarrow \mathbb{R}$ such that $h(\pm 1) = \sin h(\pm\pi/2)$ there is unique continuous function $f : [-\pi/2, \pi/2] \longrightarrow \mathbb{R}$ which extends h and satisfies $f(\sin x) = \sin f(x)$ for all $x \in [-\pi/2, \pi/2]$.

Any such f can be extended further to a continuous function on $[-\pi/2, 3\pi/2]$ by the formula $f(x) = f(\pi - x)$ for $x \in [\pi/2, 3\pi/2]$. Since $\sin(\pi - x) = \sin x$, the equality $f(\sin x) = \sin f(x)$ holds for all $x \in [-\pi/2, 3\pi/2]$. Furthermore, $f(-\pi/2) = f(3\pi/2)$. Now we can extend f to a continuous, periodic with period 2π function defined for any x by $f(x) = f(x - 2k\pi)$, where k is an integer such that $x - 2k\pi \in [-\pi/2, 3\pi/2]$. Since sin has period 2π, we easily see that this function satisfies the condition $f(\sin x) = \sin f(x)$ for all x.

Now start with any continuous function h on I such that $h(1) = h(\pi/2) = 0$ and $h(a) = 1$ for some $a \in (1, \pi/2)$. Let f be the corresponding extension to a continuous function on $\mathbb{R}$ which satisfies $f(\sin x) = \sin f(x)$ for all x and described above. Then $f(1_n) = 0$ and $f(a_n) = f(a)_n = 1_n$ for all n. Thus

$$\lim_{n\to\infty} \frac{f(1_n)}{1_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{f(a_n)}{a_n} = \lim_{n\to\infty} \frac{1_n}{a_n} = 1$$

. Since both 1_n and a_n converge to 0, the limit $\lim_{x\to 0^+} \frac{f(x)}{x}$ does not exist.

In light of the second solution to part a) the following questions seems of interest:

Question. Is there a differentiable function f such that $f(\sin x) = \sin f(x)$ and f' is not continuous at 0?

Note that if f is a continuous function such that $f(\sin x) = \sin f(x)$ for all x and a is such that $f(a) = k\pi/2$ for some odd integer k then the function

$$g(x) = \begin{cases} f(x) & \text{if } x \le a \\ k\pi - f(x) & \text{if } x > a \end{cases}$$

is also continuous and satisfies $g(\sin x) = \sin g(x)$ for all x.

Problem. Given h as in the Proposition above, describe all continuous functions f extending h to the whole real line such that $f(\sin x) = \sin f(x)$ for all x.

The following problem outlines some additional questions one may want to investigate.

Problem. Suppose f in our problem is differentiable twice. What can you say about $f''(0)$? If f has derivatives of all order, show that $f''(0) = 0$. Describe all f which are analytic functions.