Problem 1. We say that a vector in $\mathbb{R}^3$ is **positive** (**negative**) if all its coordinates are positive (resp. negative). Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^3$ such that the sum of any two of these vectors is either positive or negative. Prove that at least one of the vectors $v_1, v_2, v_3, v_4, v_1+v_2+v_3+v_4$ is either positive or negative.

Solution. Let us start by making the following straightforward observations:

1. the sum of two positive (negative) vectors is positive (negative).
2. if v is negative (positive) and $v+w$ is positive (negative) then w is positive (negative).

Let us assume that $v_1+v_2+v_3+v_4$ is neither positive nor negative (otherwise we are done). Each of the vectors $v_1+v_2, v_1+v_3, v_1+v_4$ is either positive or negative. At least two of them are of the same type and without any loss of generality we may and will assume that v_1+v_2 and v_1+v_3 are positive (after renumbering the vectors and possibly replacing each of them by its negative). Since we know that v_3+v_4 and v_2+v_4 are either positive or negative, we conclude by (1) that both are negative (otherwise we would have $v_1+v_2+v_3+v_4$ positive).

Suppose that v_1+v_4 is positive. Then v_2+v_3 must be negative by (1) and the assumption that $v_1+v_2+v_3+v_4$ is neither positive nor negative. Now $(v_1+v_2)+(v_1+v_3) = 2v_1+(v_2+v_3)$ is positive. It follows by (2) that $2v_1$ is positive, so v_1 is positive.

Similarly, if we assume that v_1+v_4 is negative then v_2+v_3 must be positive. Recall that v_3+v_4 and v_2+v_4 are negative so $2v_4+(v_2+v_3)$ is negative. Using (2) we conclude that $2v_4$ is negative, so v_4 is negative.

This completes the proof.

Second solution. Consider the full graph with vertices v_1, v_2, v_3, v_4. The edge joining v_i and v_j is colored red if v_i+v_j is positive and green if v_i+v_j is negative. If there is a triangle with edges of both colors, say v_i+v_j, v_i+v_k of one color and v_j+v_k of the other then v_i+v_j, v_i+v_k, and $-(v_j+v_k)$ have the same sign and therefore $(v_i+v_j)+(v_i+v_k)-(v_j+v_k) = 2v_i$ is either positive or negative (and so is v_i). If all triangles are monochromatic, then all edges have the same color and therefore $v_1+v_2+v_3+v_4$ is either positive or negative.

Remark (inspired by the solution provided by Maxwell T Meyers). Maxwell T Mayers noticed that actually one of v_1, v_2, v_3, v_4 must be either positive or negative (for vectors in $\mathbb{R}^3$). Indeed, our second solution tells us that this is the case unless all edges in the graph discussed in the second solution are of the same color. Suppose that they are of the same color. This means that all the sums v_i+v_j have the same sign. We may assume that they are all positive. Look at the first coordinates of our vectors. If two of them have non-positive first coordinates, then their sum is not positive, a contradiction. Thus at lest three of them have positive first coordinates. Call them v_1, v_2, v_3. By the same argument, two of these 3 vectors have positive second coordinates. Now one of the two vectors must have positive third coordinate, so it is a positive vector.

Note that the above solutions do not use the assumption that v_i are vectors in $\mathbb{R}^3$. The solutions work when v_i are vectors in $\mathbb{R}^n$ for any n, or even functions from some set X to $\mathbb{R}$ (with appropriately defined "positive" and "negative"). For example, one can use the same method to prove the following:

Exercise. Let f_1, f_2, f_3, f_4 be polynomials such that none of the polynomials f_i+f_j has a real root. Prove that one of the polynomials $f_1, f_2, f_3, f_4, f_1+f_2+f_3+f_4$ has no real roots.