Problem 6. A universe consists of an infinite collection of galaxies G_i indexed by the integers. Each galaxy G_i consists of a finite number g_i of stars. Every star in galaxy G_i is connected with every star in galaxy G_{i-1}, with every star in galaxy G_{i+1}, possibly with some stars in galaxy G_i, and with no other stars. It is known that for every sufficiently large n the number $g_{-n} + g_{-n+1} + \ldots + g_{n-1} + g_n$ does not exceed $n^{3/2}$. Each star has mass equal to the arithmetic mean of the masses of all the stars connected with it. Prove that all stars have the same mass.

Solution. The first step is to show that all stars in a given galaxy have the same mass. To this end, let T_i be the total mass of all stars in galaxy G_i. Consider a star in galaxy G_k of smallest mass m_k and suppose it is connected with exactly l other stars in G_k, whose total mass is t. Then $t \geq lm_k$ and

$$m_k = \frac{T_{k-1} + T_{k+1} + t}{g_{k-1} + g_{k+1} + l} = \frac{T_{k-1} + T_{k+1} + t - lm_k}{g_{k-1} + g_{k+1}} \geq \frac{T_{k-1} + T_{k+1}}{g_{k-1} + g_{k+1}}.$$

On the other hand, consider a star in galaxy G_k of largest mass M_k and suppose it is connected with exactly l_1 other stars in G_k, whose total mass is t_1. Then $t_1 \leq l_1 M_k$ and

$$M_k = \frac{T_{k-1} + T_{k+1} + t_1}{g_{k-1} + g_{k+1} + l_1} = \frac{T_{k-1} + T_{k+1} + t_1 - l_1 M_k}{g_{k-1} + g_{k+1}} \leq \frac{T_{k-1} + T_{k+1}}{g_{k-1} + g_{k+1}}.$$

Since $m_k \leq M_k$, comparing the above two inequalities yields

$$m_k = M_k = \frac{T_{k-1} + T_{k+1}}{g_{k-1} + g_{k+1}}$$

and consequently all stars in galaxy G_k have the same mass $m_k = (T_{k-1} + T_{k+1})/(g_{k-1} + g_{k+1})$. It follows that $T_k = g_k m_k$ and therefore

$$m_k = \frac{m_{k-1} g_{k-1} + m_{k+1} g_{k+1}}{g_{k-1} + g_{k+1}}$$

which is equivalent to

$$(m_k - m_{k+1}) g_{k+1} g_k = (m_{k-1} - m_k) g_{k-1} g_k.$$

In other words, the quantity $(m_k - m_{k+1}) g_{k+1} g_k$ is the same for all values of k. Denote this value by D. If $D = 0$, then $m_k = m_{k+1}$ for all k, i.e. all stars have the same mass.

It remains to show that $D \neq 0$ is not possible. We will do it in the case $D > 0$. If $D < 0$ the argument is similar (or, one can note that the new galactic system with $G'_i = G_{-i}$ also satisfies the conditions of the problem and has positive value of D). As $D > 0$, we have $m_0 > m_1 > m_2 > \ldots$ is a decreasing sequence of positive numbers, hence converges to some $m \geq 0$. It follows that

$$\sum_{i=1}^{N} \frac{1}{g_i g_{i-1}} = \sum_{i=1}^{N} \frac{m_{i-1} - m_i}{D} = \frac{m_0 - m_N}{D}.$$

In other words, the series $\sum_{i=1}^{\infty} \frac{1}{g_i g_{i-1}}$ converges. We will show that this contradicts the assumption that $g_{-n} + g_{-n+1} + \ldots + g_{n-1} + g_n \leq n^{3/2}$ for all sufficiently large n.

First method (after Ashton Keith). Fix $\epsilon > 0$ and consider the sets

$$S_N = \{i : 1 \leq i \leq N \text{ and } \frac{1}{g_i g_{i-1}} \geq \frac{\epsilon}{i}\}.$$

Note that if $k \notin S_N$, then $g_k g_{k-1} \geq k/\epsilon$. Using the inequality $(a+b)/2 \geq \sqrt{ab}$ we conclude that

$$\frac{g_k + g_{k-1}}{2} \geq \frac{\sqrt{k}}{\sqrt{\epsilon}}$$

for every $k \notin S_N$. Thus

$$\sum_{k=0}^{N} g_k \geq \sum_{k=1}^{N} \frac{g_k + g_{k-1}}{2} \geq \sum_{k \notin S_N, 1 \leq k \leq N} \frac{g_k + g_{k-1}}{2} \geq \sum_{k \notin S_N, 1 \leq k \leq N} \frac{\sqrt{k}}{\sqrt{\epsilon}} \geq \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{N-|S_N|} \sqrt{k}.$$

(the most right inequality follows form the fact that $\sqrt{x}$ is increasing). Now note that

$$\sum_{k=1}^{M}\sqrt{k} \geq \int_0^M \sqrt{x}dx = \frac{2}{3}M^{3/2}.$$

Thus, for all sufficiently large N we have

$$N^{3/2} \geq \sum_{k=0}^{N} g_k \geq \frac{2}{3\sqrt{\epsilon}}(N-|S_N|)^{3/2} = \frac{2}{3\sqrt{\epsilon}}\left(1-\frac{|S_N|}{N}\right)^{3/2}N^{3/2}.$$

In particular, when $\epsilon = 1/9$, we get

$$\frac{1}{2} \geq \left(1-\frac{|S_N|}{N}\right)^{3/2}$$

for all sufficiently large N. To get a contradiction, it suffices to show that for every ϵ we have $\lim_{N\to\infty}|S_N|/N = 0$. Since $|S_N|$ is a non-decreasing sequence, we have $|S_N|/N < 2|S_{2^k}|/2^k$ for any N such that $2^{k-1} < N \leq 2^k$. Thus it suffices to show that $\lim_{k\to\infty}|S_{2^k}|/2^k = 0$. Now

$$\sum_{i=1}^{2^N}\frac{1}{g_ig_{i-1}} \geq \sum_{k=1}^{N}\sum_{i\in S_{2^k}-S_{2^{k-1}}}\frac{1}{g_ig_{i-1}} \geq \sum_{k=1}^{N}\sum_{i\in S_{2^k}-S_{2^{k-1}}}\frac{\epsilon}{i} \geq \epsilon\sum_{k=1}^{N}\frac{|S_{2^k}|-|S_{2^{k-1}}|}{2^k} = -\epsilon a_1+\frac{\epsilon}{2}\sum_{k=1}^{N}\frac{|S_{2^k}|}{2^k}+\frac{\epsilon|S_{2^N}|}{2^{N+1}}.$$

Since the series $\displaystyle\sum_{i=1}^{\infty}\frac{1}{g_ig_{i-1}}$ converges, we conclude that the series $\displaystyle\sum_{k=1}^{\infty}\frac{|S_{2^k}|}{2^k}$ also converges and therefore $\lim_{k\to\infty}|S_{2^k}|/2^k = 0$. This completes the argument.

Second method. We will use the following classical inequality.

Hölder's Inequality. *If $a_0,\ldots,a_n$ and $b_0,\ldots,b_n$ are non-negative real numbers and p,q are positive real numbers such that $1/p+1/q=1$ then*

$$\sum_{i=0}^{n}a_ib_i \leq \left(\sum_{i=0}^{n}a_i^p\right)^{1/p}\left(\sum_{i=0}^{n}b_i^q\right)^{1/q}.$$

We apply this inequality for $p = 3/2$, $q = 3$, $a_i = \sqrt[3]{g_{M+i}g_{M+i-1}}$, $b_i = 1/a_i$, $n = N-M$, where $N > M$ are positive integers. We get

$$N-M+1 \leq \left(\sum_{i=M}^{N}\sqrt{g_ig_{i-1}}\right)^{2/3}\left(\sum_{i=M}^{N}\frac{1}{g_ig_{i-1}}\right)^{1/3}.$$

Let $\epsilon > 0$. Since the series $\displaystyle\sum_{i=1}^{\infty}\frac{1}{g_ig_{i-1}}$ converges, we can find M such that $\displaystyle\sum_{i=M}^{\infty}\frac{1}{g_ig_{i-1}} < \epsilon$. Also, $\sqrt{g_ig_{i-1}} \leq (g_i+g_{i-1})/2$, so $\sum_{i=M}^{N}\sqrt{g_ig_{i-1}} \leq \sum_{i=M}^{N}g_i$. Thus

$$\frac{1}{\sqrt{\epsilon}}(N-M+1)^{3/2} \leq \sum_{i=M}^{N}g_i \leq N^3$$

for all sufficiently large N. In other words, $(1-(M-1)/N) < \sqrt[3]{\epsilon}$, for all sufficiently large N. This is clearly false for any $\epsilon < 1$.

Exercise. Suppose that $t > 3/2$. Is there a galactic system which satisfies the conditions of the problem, in which not all stars have the same mass, and such that $g_{-n}+g_{-n+1}+\ldots+g_{n-1}+g_n$ does not exceed n^t for all sufficiently large n?

Exercise. Use the AM-GM inequality: $(x_1+\ldots+x_n)/n \geq \sqrt[n]{x_1\ldots x_n}$ to derive a special case of Hölder's inequality when $b_i = 1/a_i$:

$$n \leq \left(\sum_{i=1}^{n}a_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\frac{1}{a_i^q}\right)^{1/q}.$$

Hint: Use AM-GM twice.

Exercise. a) Let a, b be positive real numbers. If $p = n/k > 1$ is a rational number and $q = n/(n-k)$, use the AM-GM inequality to prove that

$$a^{1/p} b^{1/q} \leq \frac{1}{p} a + \frac{1}{q} b.$$

Conclude that the last inequality holds for all positive real numbers p, q such that $1/p + 1/q = 1$.

b) Use a) to prove Hölder's inequality. Hint: reduce first to the case when $\sum a_i^p = 1 = \sum b_i^q$.

Exercise. Prove that if $0 < \alpha < 1$ and $t > 0$ then $t^\alpha - 1 \leq \alpha(t-1)$. Use this to show the inequality in part a) of the previous problem (hint: take $t = b/a$).