Problem 6. Let $a_1 = 3/2$ and $a_{n+1} = a_n^2 - a_n + 1$. Compute the sum

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots.$$

Solution. We have

$$a_{n+1} = a_n^2 - a_n + 1 = (a_n^2 - 2a_n + 1) + a_n = (a_n - 1)^2 + a_n \geq a_n.$$

It follows that the sequence (a_n) is non-decreasing. Thus it either tends to ∞ or converges to some limit g. In the latter case, we would have $g = g^2 - g + 1$, which is equivalent to $g = 1$. However this is not possible, as $a_n \geq a_1 = 3/2$ for all n and consequently $g \geq 3/2$. Thus $\lim_{n\to\infty} a_n = \infty$.

Our solution is based on the following observation:

$$\frac{1}{a_n} + \frac{1}{a_{n+1} - 1} = \frac{1}{a_n} + \frac{1}{a_n^2 - a_n} = \frac{1}{a_n - 1}.$$

It follows that that for $n > 1$ we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{a_{n+1} - 1} = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_{n-1}} + \frac{1}{a_n - 1}.$$

In other words, the sequence

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{a_{n+1} - 1}$$

is constant. Its value for $n = 1$ is $\frac{1}{a_1} + \frac{1}{a_2 - 1} = 2$. Thus

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = 2 - \frac{1}{a_{n+1} - 1}$$

for all n. Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \lim_{n\to\infty} \left(2 - \frac{1}{a_n - 1}\right) = 2.$$

Remark. Ashton uses the same observation but slightly differently. Assume that the series converges. Since

$$-\frac{1}{a_n - 1} + \frac{1}{a_n} = -\frac{1}{a_{n+1} - 1},$$

we have

$$-\frac{1}{a_1 - 1} + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots = -\frac{1}{a_2 - 1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \dots = -\frac{1}{a_3 - 1} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \dots = \dots.$$

He then claims that this clearly implies that all the equal sums in the last line are 0. This however needs justification, which is not hard: since the series converges, we know that as n increases, $1/a_n$ tend to 0 and so does $1/(a_n - 1)$, and $r_n = 1/a_n + 1/a_{n+1} + \dots$ also tends to 0. It remains to show that the infinite series actually converges. Surprisingly, both solvers struggled with this task. We showed in our solution that a_n tends to infinity. Thus

$$\frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = \frac{a_n}{a_{n+1}} < \frac{a_n}{a_2^2 - a_n} = \frac{1}{a_n - 1}$$

tends to 0, hence the series converges by the ratio test.

Exercise. Solve the problem when $a_1 = a > 1$. What happens when $a_1 < 1$?

Problem. Prove that $a_n^{1/2^n}$ converges to some limit $l > 1$.

Analyze the reason why the above solution works and then solve the following problem.

Problem. Let $a_1 = a > 1$ and $a_{n+1} = \dfrac{a_n^2}{(\sqrt{a_n} - 1)^2}$. Compute $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$.