Problem 7. Let S be a finite set with n elements. What is the largest possible number k such that one can choose k non-empty subsets of S so that for any two of these subsets, either they are disjoint or one is contained in the other.

Solution. A collection of non-empty subsets of a set S will be called *detached* if for any two subsets in the collection, either they are disjoint or one is contained in the other. Our solution will be based on the following straightforward observation:

If F is a detached collection of subsets of a set A and G is a detached collection of subsets of a set B, and the sets A, B are disjoint, then the collection $F \cup G \cup \{A \cup B\}$ of subsets of the set $A \cup B$ is also detached.

It follows that every maximal detached collection of subsets of a set S contains S as its member (it is also easy to see that such collection contains all one-element subsets of S).

Suppose that a_n is the largest possible size of a detached collection of subsets of a set with n-elements. We are asked to compute a_n. It is easy to see that $a_1 = 1$ and $a_2 = 3$.

Consider now a set S with n elements and a detached family L of subsets of S consisting of a_n elements. We know that S belongs to the family L. Let A be a member of L different from S with largest number m of elements. Denote by B the complement $S - A$ of A in S. If $C \neq S$ is in L and C is not disjoint from A then C must be contained in A (as A can not be properly contained in C). Thus every set in L different from S is either contained in A or contained in B. The sets in L contained in A form a detached collection of subsets of A, hence we have at most a_m such sets. Likewise, the sets in L contained in B form a detached collection of subsets of B, hence we have at most a_{n-m} of them. It follows that $a_n \leq a_m + a_{n-m} + 1$. To summarize, we proved that

$$\text{there is } m \text{ such that } \ 1 \leq m < n \ \text{ and } \ a_n \leq a_m + a_{n-m} + 1. \tag{1}$$

Fix now any l such that $1 \leq l < n$. Choose a subset A of S with l elements and let $B = S - A$. According to our key observation, if F is a detached collection of subsets of a set A of size a_l and G is a detached collection of subsets of a set B of size a_{n-l}, then the collection $F \cup G \cup \{S\}$ of subsets of the set S is detached, so it has at most a_n members. This proves that

$$a_n \geq a_l + a_{n-l} + 1 \ \text{ for every } l \text{ such that } \ 1 \leq l < n. \tag{2}$$

Putting (1) and (2) together we conclude that

$$a_n = \max_{1 \leq l < n} (a_l + a_{n-l} + 1) \tag{3}$$

Playing with the formula (3) a bit leads us easily to the realization that $a_n = 2n - 1$. We can prove this by a simple induction. Indeed, it works for $n = 1, 2$. Suppose that $n > 2$ and we already know that $a_l = 2l - 1$ for all l smaller than n. Then $a_l + a_{n-l} + 1 = 2n - 1$ for all l and there fore $a_n = 2n - 1$ by (3).

Yuqiao Huang's solution (edited). We prove by induction on n that any maximal detached collection of subsets of a set with n elements consists of $2n-1$ sets. It is straightforward to check this for $n = 1, 2$. We also note that any maximal detached collection of subsets of a set S contains S and all one-element subsets of S. We claim that any maximal detached collection of subsets must also contain a set with 2 elements. Yuqiao did not explain this well, but here is an argument. Among all sets in the collection which have more than one element choose one, say K, with smallest number of elements. If K has 2 elements, we are done. If K had at least three elements, then take any subset L of K with 2 elements. Any member of our collection which has non-empty intersection with L has also a non-empty intersection with K. Hence it either contains K, and then it also contains L, or it is contained in K, but then it has only one element (by our choice of K), so it is contained in L. This shows that adding L to our collection produces again a detached collection, contrary to the assumption that the collection is maximal.

Now we assume that $n \geq 2$ and the result is true for sets with n elements. Consider a set S with $n+1$ elements and a maximal detached collection F of subsets of S. F contains a set $\{a_1, a_2\}$ with 2 elements and it also contains the sets $\{a_1\}$ and $\{a_2\}$. We may assume that $S = \{a_1, a_2, \ldots, a_{n+1}\}$. Let F_1 consist of all sets in F except the sets $\{a_1\}$ and $\{a_2\}$. Note that any set in F_1 is either disjoint from $\{a_1, a_2\}$ or contains $\{a_1, a_2\}$. Let $T = \{a_2, a_3, \ldots, a_{n+1}\}$. If $K \in F_1$ is disjoint from $\{a_1, a_2\}$ then $K = K \cap T$. If K contains $\{a_1, a_2\}$, then $K = \{a_1\} \cup (K \cap T)$. It follows that the collection $G = \{K \cap T : K \in F_1\}$ of subsets of T has the same number of elements as F_1 and it is detached. We claim that G is maximal. Otherwise we could add a subset M of T to G and still have a detached collection of subsets of T. Let $N = M$ if a_2 is not in M and $N = M \cup \{a_1\}$ otherwise. It is easy to see that adding N to F would result in a detached collection, contrary to F being maximal. Thus G is maximal and therefore has $2n-1$ elements by our inductive assumption. Since F has two more elements than F_1, we see that F has $2n - 1 + 2 = 2(n+1) - 1$ elements. This completes both our inductive step and our proof.

Remark. If $S = \{a_1, \ldots, a_n\}$ the the subsets

$$\{a_1\}, \{a_2\}, \ldots, \{a_n\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \ldots, \{a_1, a_2, \ldots, a_n\}$$

form a detached collection of $2n - 1$ subsets of S.

We end with the following slightly more challenging problem which offers a nice excursion into elementary combinatorics.

Problem. Find the number d_n of detached collections of size $2n - 1$ of a given set with n elements.
Hint. You should get $(2n-3)!!$. Recall that $m!!$ denotes the product of 1 and all positive integers not exceeding m and of the same parity as m. So $(-1)!! = 1$, $4!! = 8$, and $5!! = 15$.