Problem 6. Let M be an $m \times n$ matrix whose entries are positive real numbers. For each column of M compute the product of all the numbers in that column. Let $S(M)$ be the sum of all these products. Now let N be the matrix obtained form M by putting entries in each row in the non-decreasing order. Prove that $S(N) \geq S(M)$.

Solution. Our solution is based on the following simple, but very useful, observation.

Lemma. Suppose that $0 < a < b$ and $0 < A < B$ are real numbers. Then $aA + bB > aB + bA$.

Indeed, the result follows easily from the obvious fact that $(b - a)(B - A) > 0$.

Consider now our matrix M and let Π be the set of all matrices which can be obtained from M by permuting entries in each row. Note that if a matrix K is in Π then any permutation of the columns of K produces a matrix L which is also in Π and such that $S(K) = S(L)$. Moreover, permuting entries in each row of K produces a matrix which is again in Π.

For any matrix K let $P_i(K)$ be the product of all entries in the i-th column of K.

Choose now a matrix K in Π for which $S(K)$ is largest possible. We may assume that $P_1(K) \leq P_2(K) \leq \ldots \leq P_n(K)$ (if not, just permute the columns of K appropriately). We claim that $K = N$, i.e. that the entries in each row of K are in the non-decreasing order. Suppose otherwise. Then there is an s and $i < j$ such that the entry a in the s-th row j-th column of K is smaller that the entry b in the same row and column i: $a < b$. Let $B = P_j(K)/a$ and $A = P_i(K)/b$. Since $P_i(K) \leq P_j(K)$, we have $A < B$. Let L be the matrix obtained from K by swapping the i-th and j-th entries in the s-th row. Note that $S(L) - S(K) = (aA + bB) - (aB + bA) = (B - A)(b - a) > 0$. In other words, we have a matrix L in Π such that $S(L) > S(K)$, contrary to our choice of K. This proves that $K = N$ and therefore $S(N) = S(K) \geq S(M)$.

Remark. Note that our proof shows that a matrix in Π maximizes S if and only if it is obtained from N by permuting its columns.

Exercise. Given mn positive real numbers arrange them into an $m \times n$ matrix M so that $S(M)$ is as large as possible..