Problem 5. Positive integers $a < b < c$ are lengths of sides of a right triangle whose inradius is equal to $\gcd(a+1, b)^2$. Find a, b, c.

Solution. We start by recalling some basic concepts from elementary geometry. The *inradius* of a triangle is the radius of the circle inscribed in the triangle (called the *incircle*). The center of the incircle is called the *incenter* of the triangle. The incenter coincides with the point where all three angle bisectors of the triangle intersect.

Let O and r be the incenter and inradius of a triangle $\triangle ABC$, $a = BC, b = AC, c = AB$. In each of the triangles $\triangle AOB$, $\triangle BOC$, and $\triangle COA$ the height from vertex O is equal to r. It follows that the area of the triangle $\triangle ABC$ is equal to $ar/2 + br/2 + cr/2 = r(a + b + c)/2$.

Suppose now that the incircle is tangent to AB at C_1, to BC at A_1, and to AC at B_1. Then $AC_1 = AB_1 = x$, $BA_1 = BC_1 = y$, $CA_1 = CB_1 = z$ and $x + y = c$, $x + z = b$, $y + z = a$. Tus $z = (a + b - c)/2, y = (a + c - b)/2, x = (b + c - a)/2$.

Suppose now that the angle $\angle ACB$ is right. Then the quadrilateral CA_1OB_1 is a square. Thus $z = r = (a + b - c)/2$. Another way to see this is to look at the area: $ab/2 = r(a + b + c)/2$, i.e. $(a + b + c)r = ab$. Now

$$(a + b + c)(a + b - c) = (a + b)^2 - c^2 = a^2 + b^2 - c^2 + 2ab = 2ab.$$

This implies that $r = (a + b - c)/2$.

We are ready now for our first solution.

Solution 1 (inspired by Yuqiao Huang's solution). Let r be the inradius of our triangle, so $r = t^2$, where $t = \gcd(a + 1, b)$. Note that t divides $a + 1$, so t and a are relatively prime. Thus r and a are also relatively prime and therefore $a - 2r$ and r are relatively prime. We will need this later.

The formula for the inradius of a right triangle yields $c = a + b - 2r$. We see that $a - 2r = c - b$ is positive. From $ab = r(a + b + c)$ and $2r = a + b - c$ we get

$$(a-2r)(b-2r) = ab-2r(a+b)+4r^2 = r(a+b+c)-2r(a+b)+4r^2 = 4r^2-r(a+b-c) = 4r^2-2r^2 = 2r^2.$$

It follows that $a - 2r$ divides $2r^2$. But we have seen that $a - 2r$ and r^2 are relatively prime, hence $a - 2r$ must divide 2. It follows that $a - 2r = 1$ or $a - 2r = 2$, i.e. $(a + 1) - 2r = 2$ or $(a + 1) - 2r = 3$. Recall now that $r = t^2$ and t divides $a + 1$, so t divides $a + 1 - 2r$. This leaves us with only 4 possibilities:

- $t = 1$, $a - 2r = 1$, and $b - 2r = 2r^2$, so $r = 1$, $a = 3$, $b = 4$. However $\gcd(a + 1, b)^2 = 16 \neq 1$, so this case is not possible.
- $t = 2$, $a - 2r = 1$, and $b - 2r = 2r^2$, so $r = 4$, $a = 9$, $b = 40$. However $\gcd(a + 1, b)^2 = 100 \neq 4$, so this case is not possible.
- $t = 1$, $a - 2r = 2$, and $b - 2r = r^2$, so $r = 1$, $a = 4$, $b = 3$. However $4 > 3$, which contradicts our assumption that $a < b$, so this case is not possible.
- $t = 3$, $a - 2r = 2$, and $b - 2r = r^2$, so $r = 9$, $a = 20$, $b = 99$. It follows that $c = 101$ and this triangle satisfies the conditions of the problem.

So our problem has unique solution $a = 20, b = 99, c = 101$.

Before we state our second solution, we review some well known terminology. A right triangle in which the lengths of all three sides are integers is called a *Pythagorean triangle*. A *Pythagorean triple* consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$, i.e. it consists of side-lengths of a Pythagorean triangle. A Pythagorean triple (a, b, c) or the corresponding Pythagorean triangle are called *primitive* if the integers a, b, c do not have a common divisor greater than 1. It is easy to see that every Pythagorean triple is of the form (da, db, dc) where (a, b, c) is a primitive Pythagorean triple and d is a positive integer.

We claim that in any Pythegorean triple (a, b, c) one of the numbers a, b must be even. Indeed, if $a = 2m + 1$ and $b = 2n + 1$ were both odd then $a^2 + b^2 = 4(m^2 + n^2 + m + n) + 2$ so $a^2 + b^2$ would be even but not divisible by 4. On the other hand, $a^2 + b^2 = c^2$ so c would be even, hence $c^2 = a^2 + b^2$ would be divisible by 4, a contradiction.

If (a, b, c) is a Pythagorean triple and a is even, then either both b and c are odd, or they both are even. In any case, the number $a+b-c$ is even and therefore the inradius $r = (a+b-c)/2$ is an integer. Thus the inradius of any Pythagorean triangle is an integer. In particular, if r and a are relatively prime, then the Pythagorean triangle is primitive. Thus any triangle satisfying the conditions of our problem must be primitive.

Suppose now that (a, b, c) is a primitive Pythagorean triple and a is even. Then b and c must be both odd and relatively prime. We have

$$\left(\frac{a}{2}\right)^2 = \frac{c^2 - b^2}{4} = \frac{c-b}{2}\frac{c+b}{2}.$$

Note that $m = (c + b)/2$ and $n = (c - b)/2$ are integers and they are relatively prime. Indeed, any common divisor of m and n would also divide both $m + n = c$ and $m - n = b$, but b and c are relatively prime. Note also that m and n are of different parity, as c and b are odd. We will now use the following simple fact from elementary number theory: if the product of two relatively prime positive integers is a square then each factor is a square. Applying this to $mn = (a/2)^2$, we conclude that both m and n are squares, i.e. $m = u^2$ and $n = w^2$ for some positive integers u, w such that $a/2 = uw$. It follows that

$$a = 2uw, \quad b = u^2 - w^2, \quad c = u^2 + w^2$$

where $u > w$ are positive, relatively prime integers of different parities. Conversely, any choice of such u, w leads to a primitive Pythagorean triple (a, b, c) given by the above formulas. Note that the inradius r of the primitive Pythagorean triangle corresponding to u, w is equal to

$$r = (a + b - c)/2 = (2uw + u^2 - w^2 - u^2 - w^2)/2 = w(u - w).$$

We are ready for our second solution.

Solution 2. Suppose that (a, b, c) is a Pythagorean triple which satisfies the conditions of our problem. As we noted above, it must be primitive.

Case 1. a is even.

Then there are relatively prime positive integers $u > w$ of different parities such that

$$a = 2uw, \quad b = u^2 - w^2, \quad c = u^2 + w^2, \quad r = w(u - w).$$

Note that w divides a, so w and $a+1$ are relatively prime. On the other hand, w divides $r = \gcd(a+1, b)^2$, which in turn divides $(a+1)^2$. This means that $w = 1$ and therefore $r = u-1 = \gcd(2u+1, u^2-1)^2$. Note that $u^2-1 = (u+1)(u-1)$ and $\gcd(2u+1, u+1) = 1$. Thus $\gcd(2u+1, u^2-1) = \gcd(2(u-1)+3, u-1) = \gcd(3, u-1)$ can be either 1 or 3. In the former case, we have $u - 1 = 1$ so $u = 2$, $a = 4$, $b = 3$, $c = 5$. This however does not satisfy the assumption that $a < b$. In the latter case, $u - 1 = 9$, so $u = 10$, $a = 20, b = 99, c = 101$. It is easy to se that $(20, 99, 101)$ is indeed a solution to our problem.

Case 2. b is even.

Then there are relatively prime positive integers $u > w$ of different parities such that

$$b = 2uw, \quad a = u^2 - w^2, \quad c = u^2 + w^2, \quad r = w(u - w).$$

Note that $u - w$ divides $a = (u - w)(u + w)$, so $u - w$ and $a + 1$ are relatively prime. On the other hand, $u - w$ divides $r = \gcd(a + 1, b)^2$, which in turn divides $(a + 1)^2$. This means that $u - w = 1$ and therefore

$$r = w = \gcd(2w + 2, 2w(w + 1))^2 = 4(w + 1)^2,$$

which is not possible. Thus there are no solutions in this case.

So our problem has unique solution $a = 20, b = 99, c = 101$.

To practice the above ideas, we end with the following problems

Exercise. Find all Pythagorean triangles in which the inradius divides $(a+1)^2$, where a is the length of one of the legs of the triangle (there are 5 such triangles).

Exercise. Find all Pythagorean triangles in which the inradius is equal to 9 (there are five such triangles).