Problem 4. Let $p > 2$ be an odd prime number. Integers $a_1, a_2, \ldots, a_{p+1}$ in the interval $[0, p]$ have the following property: for every permutation π of the set $\{1, 2, \ldots, p+1\}$ the number

$$\sum_{k=1}^{p+1} k a_{\pi(k)}$$

is not divisible by p. Prove that $a_1 = a_2 = \ldots = a_{p+1}$.

Solution. First we make some observations. It suffices to show that the integers $a_1, \ldots, a_{p+1}$ are congruent to each other modulo p. Indeed, if this is so and one of them is divisible by p then all are divisible by p and this contradicts our assumption. Otherwise none of them is divisible by p and being in the interval $[0, p]$ they all must be equal to each other.

Consider now integers $b_1, \ldots, b_p$. Let B be the set of congruence classes modulo p of all numbers of the form $\sum_{k=1}^{p} k b_{\pi(k)}$ for some permutation π of the set $\{1, 2, \ldots, p\}$.

Key observation for our original solution. Let $s = b_1 + \ldots + b_p$. For any permutation π of the set $\{1, 2, \ldots, p\}$ we have $s = b_{\pi(1)} + \ldots + b_{\pi(p)}$ and

$$\sum_{k=1}^{p} k b_{\pi(k)} + s = \sum_{k=1}^{p} (k b_{\pi(k)} + b_{\pi(k)}) =$$

$$2b_{\pi(1)} + 3b_{\pi(2)} + \cdots + p b_{\pi(p-1)} + (p+1) b_{\pi(p)} \equiv b_{\pi(p)} + 2b_{\pi(1)} + 3b_{\pi(2)} + \cdots + p b_{\pi(p-1)} \pmod{p}.$$

It follows that if a congruence class of a modulo p belongs to B then the congruence class of $a + s$ also belongs to B.

Corollary 1. If s is not divisible by p then B consists all congruence classes modulo p.

Indeed, for any a the congruence classes of $a, a+s, a+2s, \ldots a+(p-1)s$ are all different, hence exhaust all congruence classes modulo p.

Key observation for Ashton Keith's solution. Recall that if m is not divisible by p then there is a permutation τ of the set $\{1, 2, \ldots, p\}$ such that

$$mi \equiv \tau(i) \pmod{p}$$

for $i = 1, 2, \ldots, p$. It follows that

$$m \sum_{k=1}^{p} k b_{\pi(k)} \equiv \sum_{k=1}^{p} \tau(k) b_{\pi(k)} = \sum_{l=1}^{p} l b_{\pi \circ \tau^{-1}(l)} \pmod{p}$$

for any permutation π of the set $\{1, 2, \ldots, p\}$ (here $\pi \circ \tau^{-1}$ is the composition of the permutation π and the permutation τ^{-1}, the inverse of τ; this comes from setting $\tau(k) = l$, so $k = \tau^{-1}(l)$). It follows that if a congruence class of a belongs to B then the congruence class of ma also belongs to B.

Corollary 2. If B contains a non-zero congruence class then B contains all non-zero congruence classes modulo p.

Indeed, if a is not divisible by p then the congruence classes of $a, 2a, \ldots (p-1)a$ exhaust all non-zero congruence classes modulo p.

Now we are ready for our solutions.

Solution 1. Suppose that the numbers $a_1, \ldots, a_{p+1}$ are not all congruent to each other modulo p. We will show that this leads to a contradiction. First we claim that we can choose p among our numbers whose sum is not divisible by p. Indeed, if a_i and a_j are not congruent modulo p, then $(a_1 + \ldots + a_{p+1}) - a_i$ and $(a_1 + \ldots + a_{p+1}) - a_j$ are also not congruent modulo p, hence one of them is not divisible by p. Choose such p numbers and call them $b_1, ..., b_p$. Call the remaining number from our set b_{p+1}. Since $b_1 + \ldots + b_p$ is not divisible by p, Corollary 1 tells us that the congruence class of $-(p+1)b_{p+1}$ belongs to B, i.e. there is a permutation π of the set $\{1, 2, \ldots, p\}$ such that

$$\sum_{k=1}^{p} k b_{\pi(k)} \equiv -(p+1) b_{p+1} \pmod{p}.$$

In other words, $\sum_{k=1}^{p} kb_{\pi(k)} + (p+1)b_{p+1}$ is divisible by p and the sequence $b_{\pi(1)}, \ldots, b_{\pi(p)}, b_{p+1}$ is a permutation of $a_1, \ldots, a_{p+1}$, a contradiction.

Solution 2, due to Ashton Keith. Suppose that the numbers $a_1, \ldots, a_{p+1}$ are not all congruent to each other modulo p. We will show that this leads to a contradiction. We consider two cases.

Case 1. Among the numbers $a_1, \ldots, a_{p+1}$ exactly one is not divisible by p.

Then there is a permutation π of the set $\{1, 2, \ldots, p+1\}$ such that only $a_{\pi(p)}$ is not divisible by p. Thus $\sum_{k=1}^{p+1} ka_{\pi(k)}$ is divisible by p, a contradiction.

Case 2. Among the numbers $a_1, \ldots, a_{p+1}$ at least two are not divisible by p.

Note that we may choose one of these two numbers so that not all of the remaining numbers are congruent to each other. In other words, we can reorder our numbers as $b_1, ..., b_{p+1}$ so that b_{p+1} is not divisible by p and among $b_1, ..., b_p$ there are at least two non-congruent modulo p numbers. We may assume that b_1 and b_2 are not congruent to each other. It follows that one of the numbers $b_1 + 2b_2 + \ldots + pb_p$ and $b_2 + 2b_1 + 3b_3 + \ldots + pb_p$ is not divisible by p. Thus the set B contains a non-zero congruence class. By Corollary 2, B contains all non-zero congruence classes. It follows that there is a permutation π of the set $\{1, 2, \ldots, p\}$ such that

$$\sum_{k=1}^{p} kb_{\pi(k)} \equiv -b_{p+1} \equiv -(p+1)b_{p+1} \pmod{p}.$$

In other words, $\sum_{k=1}^{p} kb_{\pi(k)} + (p+1)b_{p+1}$ is divisible by p and the sequence $b_{\pi(1)}, \ldots, b_{\pi(p)}, b_{p+1}$ is a permutation of $a_1, \ldots, a_{p+1}$, a contradiction.

To practice the above ideas, we end with the following problem from a high school mathematical competition in Poland.

Exercise. Let $p > 2$ be a prime. Given a set of $p+1$ distinct integers, prove that one can choose $p-1$ among them, $a_1, \ldots, a_{p-1}$ such that $\sum_{k=1}^{p-1} ka_k$ is divisible by p.