Problem 2. Let $\mathbb{N}_0$ be the set $\{0, 1, 2, \ldots\}$ of all non-negative integers. Find all functions $f : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ such that $f(a^2 + b^2) = f(a)^2 + f(b)^2$ for all a, b in $\mathbb{N}_0$.

Solution. Let us start by taking $b = 0$ in the formula $f(a^2 + b^2) = f(a)^2 + f(b)^2$. We get $f(a^2) = f(a)^2 + f(0)^2$. Taking $a = 0$ yields $f(0) = f(0)^2 + f(0)^2$, so $f(0) = 0$ (as it is an integer). Thus $f(a^2) = f(a)^2$ for all a. For $a = 1$ we get $f(1) = f(1)^2$ so $f(1) = 0$ or $f(1) = 1$.

Note that:

$$f(2) = f(1^2 + 1^2) = 2f(1)^2;$$

$$f(4) = f(2)^2;$$

$$f(5) = f(2^2 + 1^2) = f(2)^2 + f(1)^2;$$

$$f(8) = f(2^2 + 2^2) = 2f(2)^2.$$

Now $f(5)^2 = f(25) = f(4^2 + 3^2) = f(4)^2 + f(3)^2$ so

$$f(3)^2 = f(5)^2 - f(4)^2.$$

Furthermore,

$$f(10) = f(1 + 9) = f(1)^2 + f(3)^2.$$

Since $f(10)^2 = f(100) = f(36 + 64) = f(6)^2 + f(8)^2$, we have

$$f(6)^2 = f(10)^2 - f(8)^2.$$

This tells us that if $f(1) = 0$ then $f(0) = f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = f(8) = f(10) = 0$. If $f(1) = 1$ then $f(k) = k$ for $k = 0, 1, 2, 3, 4, 5, 6, 8, 10$.

Note that we have the following identity:

$$(2n + 1)^2 + (n - 2)^2 = (2n - 1)^2 + (n + 2)^2.$$

It follows that for any $n \geq 3$ we have

$$f(2n + 1)^2 = f(2n - 1)^2 + f(n + 2)^2 - f(n - 2)^2. \tag{1}$$

For $n = 3, 4$ we get $f(7) = f(9) = 0$ if $f(1) = 0$ and $f(7) = 7$, $f(9) = 9$ if $f(1) = 1$.

Finally note the following identity:

$$(2n)^2 + (n - 5)^2 = (2n - 4)^2 + (n + 3)^2.$$

It follows that for $n \geq 6$ we have

$$f(2n)^2 = f(2n - 4)^2 + f(n + 3)^2 - f(n - 5)^2. \tag{2}$$

Now we use induction on k to show that $f(k) = 0$ for all k if $f(1) = 0$ and $f(k) = k$ for all k if $f(1) = 1$. Indeed, for $k \leq 10$ we have already verified this claim. Let $k > 10$ and assume that the result holds for all non-negative integers smaller than k. When $k = 2n + 1$ is odd then the result holds for k by (1). When $k = 2n$ is even, the result holds for k by (2).

Thus there are only 2 functions satisfying the conditions of the problem: $f(n) = 0$ for all n or $f(n) = n$ for all n.

Challenge. Solve the problem when $\mathbb{N}_0$ is replaced by the set $\mathbb{N}$ of all positive integers (note that without 0 there is no simple way to deduce $f(a^2) = f(a)^2$ and compute $f(1)$).