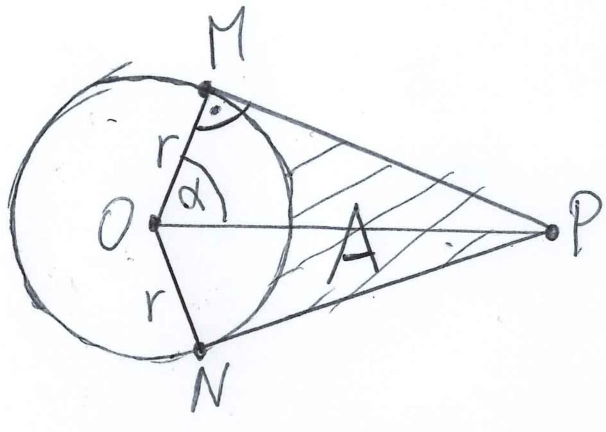
Problem 1. A loop of string has fixed length L. It is looped around a disk of radius r and pulled tight at one point so as to form an "ice cream cone" shape as pictured below. Consider the region labeled A that is inside the loop of string, but outside the disk. Note that the area of A is zero if either $r = 0$ or if $r = L/2\pi$. What value of r maximizes the area of the region A and what is this maximum value of the area?



Solution. The area S of the region A is the difference between the area S_1 of the quadrilateral $OMPN$ and the area S_2 of sector MON of the circle. The area S_1 is just twice the area of the traingle OMP. Note that OMP is a right triangle as the line MP is tangent to the circle. Thus $MP = NP = r\tan\alpha$ and $S_1 = r \cdot MP = r^2 \tan\alpha$. Now $S_2 = \pi r^2(2\alpha/2\pi) = r^2\alpha$ so

$$S = r^2(\tan\alpha - \alpha).$$

Note that the string is the union of the segments MP and NP and the "long" arc with ends M, N (corresponding to the central angle $2\pi - 2\alpha$). Thus $L = 2MP + r(2\pi - 2\alpha) = 2r\tan\alpha + 2r(\pi - \alpha) = 2r(\tan\alpha - \alpha) + 2\pi r$. It follows that

$$\tan\alpha - \alpha = \frac{L}{2r} - \pi.$$

Thus we get

$$S = S(r) = r^2\left(\frac{L}{2r} - \pi\right) = \pi r\left(\frac{L}{2\pi} - r\right).$$

The problem asks to find $r \in (0, L/2\pi)$ for which $S(r)$ is largest. This is now a simple calculus question. The maximum is at $r = L/4\pi$ and it is equal to $S_{max} = L^2/16\pi$.