Chapter 1

Algebra Review

If you have not yet read "Dear Student" at the beginning of the text, please do so. Now would be a good time.

Good. We're all set. Pick a number between 1 and 10. Write it down. This is your first test to see if you are really going to follow the directions in this book. Have you got your number written down? Don't erase it. We'll come back to it later.

1.1 Terms, Definitions, Notation

1.1.1 The Real Number System

This course will be dealing only with real numbers. Most of your experience with mathematics so far has likely been within the framework of the Real Number system. Perhaps you have sometime in the past touched on the set of complex numbers, which is built around having a value, called "i," for $\sqrt{-1}$. We will not be working with this. In the real number system $\sqrt{-1}$ is undefined and so that is how we will regard it in this course.

Real numbers are those that you would encounter and find meaningful in everyday life. A very basic definition for a real number is:

Definition 1.1.1.
A real number is a number that can be expressed as a decimal.

Notice that the definition says that a real number *can* be written as a decimal, not that it must be. For example, $\sqrt{2}$ is a real number and $\frac{1}{3}$ is a real number even though they are expressed here in non-decimal form.

$\mathbb{R}$

We will use the symbol $\Re$ to represent the Set of Real Numbers.

There are some important subsets of $\Re$ with which you need to be familiar.

Definition 1.1.2.
An integer is a whole number, positive, negative or zero. The set of integers, then is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. *We will use the symbol* $\mathbb{Z}$ *to represent the Set of Integers.*

Definition 1.1.3.
A rational number is a number that can be expressed as a quotient of integers. We will use the symbol $\mathbb{Q}$ *to represent the Set of Rational Numbers.*

Again, notice the use of the word *can*. Examples of rational numbers are: $\frac{3}{5}$ and $-\frac{12}{11}$. Other examples are $.\overline{3}$ (which *can* be written as $\frac{1}{3}$) and .75 (which *can* be written as $\frac{3}{4}$), and 5 (which *can* be written as $\frac{5}{1}$). Indeed from the last example, it is easy to see that the set of integers is a subset of the set of rational numbers. $\mathbb{Z} \subset \mathbb{Q}$.

Definition 1.1.4.
An irrational number is a real number that is not rational. It cannot be written in the form $\frac{a}{b}$ *where* a *and* b *are integers. Examples of irrational numbers are:* $\sqrt{2}$, $\sqrt{5}$ *and* π.

From the way we defined irrational numbers it is clear that $\mathbb{R}$ can be divided into two distinct subsets, the rational numbers and the irrational numbers. Given any decimal representation of a real number, how can we tell if the number is rational or irrational?

- We have already said that all integers are rational.
- If a decimal expression terminates at some point then the number is rational. This can be seen by a simple conversion of the decimal into its fractional equivalent. $.71 = \dfrac{71}{100}$, $23.9721 = \dfrac{239,721}{10,000}$.
- If a decimal expression does not terminate but at some point has a pattern that is repeated from thence on, then the number is rational. $.\overline{4} = \dfrac{4}{9}$, $.\overline{567} = \dfrac{567}{999}$, $.10\overline{7} = \dfrac{97}{900}$. You probably learned at some time how to convert these numbers, but it wouldn't be unreasonable if you were rusty. This technique is reviewed in the exercises.
- A decimal that does not terminate and that never reaches the point of having a repeating pattern is irrational.

This last item isn't always entirely helpful if we have a non-decimal representation of a number. How do we know that $\sqrt{2}$ does not have a repeating pattern in its decimal expansion? How do we know that π does not? This is not a simple question to answer in general. There are proofs that verify that these two numbers are in fact irrational. The proof for the irrationality of $\sqrt{2}$ can be generalized to say that if p is a prime number[1] then $\sqrt{p}$ is irrational. We will not go into these proofs here. Just accept that these are examples of irrational numbers.

Comprehension Check 1.1.
Circle the integers. Put a box around the irrational numbers.

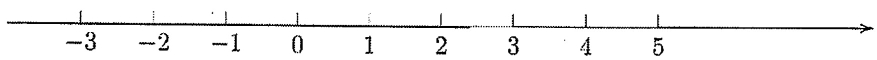
$$\frac{3}{5} \qquad 0 \qquad \sqrt{5} \qquad \pi \qquad 6.24 \qquad -18 \qquad \frac{10}{5}$$

$$(3+\sqrt{7}) \qquad 4.5\overline{8} \qquad \sqrt{16} \qquad -\frac{3}{4} \qquad \frac{\pi}{2} \qquad \left(\frac{34}{81}+\frac{17}{11}\right)$$

In the Comprehension Check you should have found four integers and four irrational numbers. How many rational numbers are there?

[1]A *prime* number is an integer greater than 1 whose only positive integer factors are itself and 1.

One important property of the set of real numbers, $\Re$, is that the numbers are ordered. If we were to line them up, each one would have a specific place relative to all of the others. In fact we do often find it convenient to think of them collectively as making up the Real Number Line. At some place on this line we indicate the position of the number zero. To the right of zero we put the positive numbers, increasing in size from zero. To the left of the zero point we put the negative numbers, decreasing from zero. The number line below is marked with integers but any real numbers can be used to label points on a number line.



We say that the real numbers "go to" infinity (∞) or minus-infinity ($-\infty$). This language can cause some misconceptions.

needs definition

Important Idea 1.1.1.
Infinity is NOT a real number. In the context of the real number system infinity is simply a concept ~~that indicates unboundedness.~~ "∞" or "$+\infty$" indicate unboundedness in the positive direction and "$-\infty$" indicates unboundedness in the negative direction.

Important Idea 1.1.2.
Between any two real numbers is another real number.

This last idea can be expanded to conclude that there are no gaps in the real number line. (Can you justify this conclusion?) The set of points representing the ordered set of real numbers is so dense that the points form a line. Indeed there are infinitely many points between any two real numbers. So, how many numbers are there between 1 and 10? This leads us to a little math humor:

Mathematician to Student: Pick a number between 1 and 10.
Student: (fill in the number you chose at the beginning of the chapter).
Mathematician ponders: Hmmmm, an integer,...how unlikely!

Of course this "joke" falls apart if the student actually picks a non-integer value, but it is successful a very large percentage of the time.[2]

We will often want to express intervals of numbers, such as "the numbers between 1 and 10." There are three ways that we will do this: graphically (on a number line); algebraically; and with interval notation. In all instances we need to be clear whether or not we are including the endpoints, in this case 1 and 10.

1. If we DO NOT include the endpoints our notation indicates this by using:
   - open circles at 1 and at 10 on the number line:

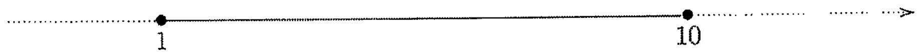
   - strict inequalities: $1 < x < 10$
   - parentheses next to the 1 and 10: $(1,\ 10)$.

   This is called an *open* interval.

[2] When our son was about eight years old and he first learned about the number π he was very excited. For a while after that whenever he was asked to "pick a number between..." and the interval requested included π, then "π" was his response. However, he soon learned that this was not helping him socially and he abandoned that tactic.

2. If we DO include the endpoints our notation indicates this by using:

   - closed circles at 1 and at 10 on the number line:



   - inclusive inequalities: $1 \leq x \leq 10$
   - brackets next to the 1 and 10: $[1,\ 10]$.

   This is called a *closed* interval.

3. If we wish to include one end but not the other we use the appropriate notation on each end. So, for example to include the 1 but not the 10 we would have:

   - closed circle at 1 but open circle at 10 on the number line:

   - inclusive inequality at 1 and strict inequality at 10: $1 \leq x < 10$
   - bracket next to the 1 and parenthesis next to the 10: $[1,\ 10)$.

   This is called a *half-open* interval.

Sometimes we will wish to express an unbounded interval such as "all of the numbers greater than 3." On the number line we would put an open circle on 3 and then shade everything to the right. In algebraic notation we would write: $x > 3$. (Be careful...this is not the same as $x \geq 4$. Why not?) In interval notation we would write: $(3,\ \infty)$. Always use parentheses rather than brackets for ∞ or $-\infty$. This emphasizes that these are not numbers that we are including, but that these are symbols indicating unboundedness. The interval is considered *open* on any end where it is unbounded.

Sometimes we will want to indicate a subset of $\Re$ that is not an interval but a set of isolated points. We can do this by listing the numbers or by showing a pattern for the numbers. These sets are enclosed in braces. For example, $\{2, 3, 6, 12\}$ is a subset of $\Re$ which contains just the four numbers listed. If we wanted to write the set of even integers we could write: $\{\ldots, -2, 0, 2, 4, \ldots\}$.

Sometimes we will want to indicate a subset of $\Re$ that is a combination of intervals or isolated points. We use the "union" symbol $\cup$ to indicate that our subset of $\Re$ is to include all of the numbers in all of the sets combined. For example if we want to write the set that includes all negative numbers and all even positive integers we would write: $(-\infty,\ 0) \cup \{2, 4, 6, 8, \ldots\}$.

Interval notation for $\Re$ is $(-\infty,\ \infty)$.

Many times in the course you will be asked to express things on a number line. It is not usually essential that the placement of points be to scale. However, you do need to put them in the correct order. To this end it is useful to remember that $\pi \approx 3.14$ and $\sqrt{2} \approx 1.41$ and $\sqrt{3} \approx 1.73$.

1.1.2 Algebraic Expressions

It is assumed that you have some experience with algebra. In this section we wish to briefly review some vocabulary. We do it non-rigorously:

- A *variable* is an entity which can take on more than one value in an expression or equation. When an *independent variable* takes on a particular value it affects the value of a *dependent variable.* Variables are usually represented by alphabetic characters. They may be letters that are associated with the meaning of the variable, such as t for time or h for height. Non-specific variables usually use letters from the end of the alphabet, often x for an independent variable and y for a dependent variable.

- A *constant* is an entity which does not change in an expression or equation. It can be represented by a number, such as "3", which is obviously fixed. Sometimes it is represented by an alphabetic character. This is an *arbitrary constant.* We may not know the exact value of an arbitrary constant, but it is fixed. It does not vary within the context of the expression or equation.

Example 1.1.1.
Suppose you have a cell phone and each month you have to pay a fixed service charge plus 10 cents per minute for usage. We could write an equation to calculate your bill which looks like this: $B = (.10)m + S$ *where* B *represents the bill (amount you have to pay, in dollars),* m *represents the number of minutes you used and* S *represents the fixed service charge. In this equation* S *is an arbitrary constant. It is constant because it does not change to vary the bill. We don't know what the amount is, it could be anything (hence "arbitrary"), but it does not change within the context of the example. On the other hand, the amount of minutes used does change from month to month so* m *is a variable. The change in* m *affects the outcome of the bill,* B*. Since* B *changes, it is a variable. Since the value of* B *is determined by the value of* m*, we know that* m *is the independent variable and* B *is the dependent variable. The number .10 is, of course, a constant.*

- A *term* is a piece of an algebraic expression that is separated from other pieces by "+". In the cell phone example we consider two algebraic expressions: "B" and "$(.10)m + S$." B is a term, $(.10)m$ is a term and S is a term.

- A *coefficient* is a constant (arbitrary or known) multiplier for a variable. In the example we have 1 as the coefficient for B and $(.10)$ as the coefficient for m.

- *Like terms* are terms in an algebraic expression or equation which are identical except possibly for their corresponding coefficients. Constant terms are always considered like terms. The example above has no like terms.

- A *factor* is any algebraic expression that is multiplied times another. $(.10)$ and m are factors within the same term.

Example 1.1.2.
Consider the equation $y = (2x^3 + 4)(x^2 + 6x - \sqrt{2}x - 1)$. *We assume that* x *is the independent variable and* y *is the dependent variable. The constants are all numbers. The coefficients are* 2, 1, 6, *and* $-\sqrt{2}$. *The terms are* y, $2x^3$, 4, x^2, $6x$, $-\sqrt{2}x$ *and* -1. *The like terms are* $6x$ *and* $-\sqrt{2}x$ *and also* 4 *and* -1. *The factors are* $(2x^3 + 4)$ *and* $(x^2 + 6x - \sqrt{2}x - 1)$. *Within terms,* 2 *and* x^3 *are factors,* 6 *and* x *are factors,* $-\sqrt{2}$ *and* x *are factors. Usually we don't bother going to the term level to identify factors. Factors will be discussed more later in the chapter.*

Notation Tips

The following expressions are all equal: $-\frac{a}{b}$ $\frac{-a}{b}$ $\frac{a}{-b}$.
Why is this true? To help you see this, evaluate each of the three expressions using $a = 6$ and $b = 3$.

We usually use alphabetic characters to represent variables. What would we do if we had an equation with more than 26 variables in it? We could use Greek letters, or upper case letters or many other symbols. But this would be quite cumbersome. Indeed using more than four or five letters in an equation can make it difficult to follow. One way to get around this is the use of subscripts. If we have four variables we could call them x_1, x_2, x_3 and x_4. This can work just as easily as using a, b, c, d. Sometimes subscripted variables can help us keep better track of what we are doing. If we want to talk about points on a graph we could talk about (x, y), (u, v) and (t, s) but it would be much more descriptive to talk about $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3). With the subscript notation we have all of our first coordinates identified as x's (instead of x, u and t) and all of our second coordinates identified as y's. Further, the subscript tells us which point we are talking about so that if we later have an equation involving y_2, we know that that is the y-coordinate that goes with the specific x-coordinate x_2. Do not let the subscripts confuse you. They do not have any arithmetic meaning: $x_2 + x_3 \neq x_5$.

There will be many times in reading this book when you come across the written expressions $x > 0$ or $x < 0$. This is concise mathematical language. Sometimes it will be helpful to your understanding of the topic if you mentally translate these into "x is positive" or "x is negative." Certainly those English statements are exactly the same as the mathematical ones but sometimes the English gives more insight into what is going on.

1.2 Working with Exponents

In this section we wish to fairly thoroughly review working with exponents. We will begin with integer exponents.

1.2.1 Integer Exponents

Our understanding of the meaning of *positive* integer exponents is that the exponent tells how many times we are to multiply the base times itself. This gives us: $3^2 = 3 \cdot 3 = 9$ and $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$. In general we have $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}$.

To multiply terms with like bases it makes sense to add the exponents: $3^2 \cdot 3^4 = 3^{2+4} = 3^6$ because $(3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$.

A general rule then is: $a^n a^m = a^{n+m}$.

To divide terms with like bases it makes sense to subtract the exponents: $\dfrac{2^5}{2^3} = 2^{5-3} = 2^2$ because $\dfrac{2^5}{2^3} = \dfrac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}$ which reduces to $\dfrac{2 \cdot 2}{1} = 2^2$.

A general rule then is: $\dfrac{a^n}{a^m} = a^{n-m}$, for $a \neq 0$.

Why do we have to say that $a \neq 0$ for the division rule, but not for the multiplication?

Important Idea 1.2.1.

Division by zero is <u>*never*</u> *allowed.*

A fraction is a division problem where the denominator is the divisor. So you can never allow a zero to be in the denominator of a fraction. When making a rule that contains a fraction we should explicitly exclude from the rule any values that would make the denominator zero.

Suppose we had $\frac{a^n}{a^n}$. By our general division rule that would be equal to $a^{n-n} = a^0$. But we know that any fraction where the numerator and the denominator are the same must have the value 1. So, we can conclude from ~~this that~~ $a^0 = 1$. Again, we must be very careful to exclude from this rule the value $a = 0$. Indeed, 0^0 is undefined.

Suppose we had $\frac{a^3}{a^5}$ ~~where $a \neq 0$~~. By our general division rule that would be equal to a^{-2}. We would want the expression $\frac{a^3}{a^5}$ to be equal to $\frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^2}$. So, we can conclude from this that $a^{-2} = \frac{1}{a^2}$.

This suggests a definition for negative exponents:

Definition 1.2.1.

$$a^{-n} = \frac{1}{a^n}, \textit{ where } a \neq 0.$$

a^n

Often this is useful when the negative exponent is in the denominator: $\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = a^n$. Essentially we discover that when a variable has a negative exponent it can be changed to a positive exponent if we change the base to its reciprocal. A more pedestrian way to describe this is that we can change the sign of the exponent if we move the base to the opposite side of the fraction line.

Example 1.2.1.

1. $a^7a^4 = a^{11}$ $\qquad b^4b^{-9} = b^{-5}$
2. $\frac{a^7}{a^{11}} = a^{-4}$ $\qquad \frac{b^4}{b^{-9}} = b^{13}$
3. $a^{-5} = \frac{1}{a^5}$ $\qquad \frac{1}{a^3} = a^{-3}$ $\qquad \frac{1}{a^{-2}} = a^2$ $\qquad 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ $\qquad \frac{1}{4^{-2}} = 16$

We can use the idea above that a^n means $\underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$ to show that the following rules are true when n is a positive integer. The first is done for you. You should verify the other two.

$(ab)^n = a^nb^n$.

$$(ab)^n = \underbrace{(ab) \cdot (ab) \cdot \ldots \cdot (ab)}_{n \text{ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} \cdot \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} = a^nb^n$$

$(a^n)^m = a^{nm}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. $\qquad$ This last expression can be written as a^nb^{-n}

A special case of the last rule, where the numerator is 1, can be written: $\left(\frac{1}{a}\right)^n = \frac{1^n}{a^n} = \frac{1}{a^n}$, which can be written as a^{-n}.

We now list all of the rules together as one Important Idea. It turns out that all of the rules for positive integer exponents will hold for any real number exponents. If later you get confused when dealing with negative, fractional or irrational exponents you can try to remember the rules by creating simple positive integer examples and then working accordingly.

Important Idea 1.2.2. *Rules for Exponents*

1. $a^n a^m = a^{n+m}$
2. $\frac{a^n}{a^m} = a^{n-m}$
3. $a^{-n} = \frac{1}{a^n}$, *where* $a \neq 0$
4. $(ab)^n = a^n b^n$
5. $(a^n)^m = a^{nm}$
6. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} = a^n b^{-n}$, *where* $b \neq 0$
7. $\left(\frac{1}{a}\right)^n = \frac{1}{a^n} = a^{-n}$, *where* $a \neq 0$

We have not so far put any restrictions on the values for the bases a or b in these rules except when one appears in the denominator of a fraction. In fact these bases can be any real number. Be careful not to make a common notational error when dealing with negative bases. Consider the following Important Idea.

MOST IMPORTANT IDEA

Important Idea 1.2.3.

The expressions $-a^n$ and $(-a)^n$ are not the same.

Of course there are some values of a and n where these expressions are the same, but in general they are not. In the expression $-a^n$ only the a is raised to the n^{th} power. In the expression $(-a)^n$, the base is $(-a)$.

Example 1.2.2.

$-3^2 = -(3 \cdot 3) = -9 \qquad (-3)^2 = (-3)(-3) = 9$

It is important to know what the base of your exponent is.

Example 1.2.3.

Rewrite so that all exponents are positive. Simplify.

1. $(3x)^2 = (3x)(3x) = 9x^2$
2. $(-2y)^3(3y^4) = (-8y^3)(3y^4) = -24y^7$
3. $(z+2)^{-3}(z+2)^{-1} = (z+2)^{-4} = \frac{1}{(z+2)^4}$

In the last example, notice that the base is $(z+2)$. We cannot split that sum. It is a matter of the order of operation. Consider the following example:

Example 1.2.4.
Rewrite so that all exponents are positive. Simplify.

1. $(3+4)^2 = 7^2 = 49,\quad$ *whereas* $\quad 3^2+4^2 = 9+16 = 25$

2. $\left(\frac{1}{3}\right)^{-1} + \left(\frac{1}{4}\right)^{-1} = 3+4 = 7,\quad$ *whereas* $\quad \left(\frac{1}{3}+\frac{1}{4}\right)^{-1} = \left(\frac{7}{12}\right)^{-1} = \frac{12}{7}$

Example 1.2.5.
Rewrite so that all exponents are positive. Simplify.

1. $\dfrac{25y^8}{10y^4} = \dfrac{5\cdot 5y^8}{2\cdot 5y^4} = \dfrac{5y4}{2}$

2. $\dfrac{b^{p+2q}}{(b^2)^q} = \dfrac{b^{p+2q}}{b^{2q}} = b^p$ $\qquad \dfrac{b^p\cdot b^{2q}}{b^{2q}} = b^p$

Example 1.2.6.
Rewrite so that all exponents are positive. Simplify.

1. $\dfrac{3x^2z^{-1}y^5}{2xzy^{-3}} = \dfrac{3x^2y^5y^3}{2xzz} = \dfrac{3xy^8}{2z^2}$

2. $\left(\dfrac{6x^{-1}y}{y^3}\right)^{-2} = \left(\dfrac{6}{xy^2}\right)^{-2} = \left(\dfrac{xy^2}{6}\right)^2 = \dfrac{x^2y^4}{36}$

3. $\dfrac{(a^{-2}b^{-1}c^3)^{-2}}{(a^3b)^3c^{-5}} = \dfrac{a^4b^2c^{-6}}{a^9b^3c^{-5}} = a^{-5}b^{-1}c^{-1} = \dfrac{1}{a^5bc}$

Example 1.2.7.
Use ~~the Rules~~ for Exponents to simplify the numbers below without using a calculator:

1. $\dfrac{2^{12}\cdot 5^{13}}{10^{12}} = \dfrac{2^{12}\cdot 5^{13}}{(2\cdot 5)^{12}} = \dfrac{2^{12}\cdot 5^{13}}{2^{12}\cdot 5^{12}} = 5$

2. $\dfrac{120}{48} = \dfrac{5\cdot 3\cdot 2^3}{3\cdot 2^4} = \dfrac{5}{2}$

1.2.2 Non-integer Exponents

? Not all exponents are integers. Some are other rational numbers; some are irrational numbers. We don't have meaning for $a^{\frac{1}{3}}$ like we have for $a^3 = a\cdot a\cdot a$. We can't talk about "a multiplied times itself $\frac{1}{3}$ times." However, as we used the Rules for Exponents to give some meaning to negative exponents we can use these same rules to get some meaning for rational exponents.

Discuss

We know from our Rules of Exponents that $(a^n)^m = a^{nm}$. Suppose m is an integer and n is its reciprocal, $\frac{1}{m}$. Then $(a^n)^m = a^{nm}$ would look like $(a^{\frac{1}{m}})^m = a^{\frac{1}{m}\cdot m}$. But this last exponent is just 1, so we are getting $(a^{\frac{1}{m}})^m = a^1 = a$. What does that tell us about the number $a^{\frac{1}{m}}$? For help answering this let's look at the specific example where $m = 3$.

Example 1.2.8.

$$(a^{\frac{1}{3}})^3 = a^{\frac{1}{3}\cdot 3} = a^1 = a$$

What does this tell us about the number $a^{\frac{1}{3}}$? It says that if we cube that number we get a. What number must it then be? The only number which when you cube it you get a is the number $\sqrt[3]{a}$. So, we conclude that $a^{\frac{1}{3}} = \sqrt[3]{a}$.

We generalize this result into the following definition:

Definition 1.2.2.
For n a positive integer, $a^{\frac{1}{n}} = \sqrt[n]{a}$. If n is even then $a \geq 0$.

We must make the stipulation that "if n is even then $a \geq 0$" because even roots (square roots, fourth roots, etc.) are not defined for negative numbers. $\sqrt{-9}$ is undefined in $\Re$.

MOST IMPORTANT

We also need to make another careful distinction when dealing with roots (radicals). By convention, when we write $\sqrt{25} = x$ our solution is $x = 5$. We are only interested in the *positive* number whose square is 25. However, if the original equation is given as $x^2 = 25$, then we have two solutions, $x = 5$ and $x = -5$. The distinction is that in the first case we were *given* a square root problem to solve; in the second case we *took* a square root to solve a different problem. This is an issue for all even roots. It is not an issue for odd roots because odd roots only have one solution anyway.

NOT WELL SAID

Important Idea 1.2.4.
When given a problem in the form $x = \sqrt[n]{a}$ we only accept one solution (one x such that $x^n = a$), and the sign ($\pm$) of x is always the same as the sign of the radicand[3] a.

Example 1.2.9. Consider only $\sqrt[4]{2^4}$ or $\sqrt[4]{4^2}$. ~~Only~~ Not $\sqrt[4]{(-2)^4}$

DO IN CLASS

1. $x = \sqrt[4]{16}$ *has one solution,* $x = 2$, *but* $x^4 = 16$ *has two solutions,* $x = \pm 2$.
2. $x = \sqrt[3]{8}$ *has one solution,* $x = 2$, *and* $x^3 = 8$ *has one solution,* $x = 2$.
3. $x = \sqrt[3]{-8}$ *has one solution,* $x = -2$, *and* $x^3 = -8$ *has one solution,* $x = -2$.
4. $x = \sqrt[4]{-16}$ *has no solution; it is undefined.* $x^4 = -16$ *has no solution.*

Fractional exponents essentially follow the same Rules for Exponents (Important Idea 1.2.2) that we developed for integer exponents, but we have to be alert. We need to be extra careful that at no step do we do something illegal. Specifically, we cannot have a negative value as the radicand for an even root. Consider the following situation:

REVIEW BEFORE TEST

According to the Rules for Exponents, $a^{\frac{n}{m}} = (a^{\frac{1}{m}})^n$ and $a^{\frac{n}{m}} = (a^n)^{\frac{1}{m}}$. When we rewrite these in radical form we see that $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ and $a^{\frac{n}{m}} = \sqrt[m]{a^n}$. This tells us that we have two ways to think about $a^{\frac{n}{m}}$. It also tells us that $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$. But, look closely at $\sqrt{x^4}$. Here, x can take on negative values because we raise the x value to the fourth power (making it positive) before we take the square root. However, when you look at $(\sqrt{x})^4$ you see that x cannot take on negative values because the action of taking the square root occurs before the raising to the fourth power. So, we can only say that $(\sqrt{x})^4 = \sqrt{x^4}$ if we stipulate that $x \geq 0$. In general then, we interpret $a^{\frac{n}{m}}$ as valid only when $a^{\frac{1}{m}}$ is valid.

See example problem 1.2.11(3) $(z^2)^{1/4}$ is valid only if $z^{1/4}$ is valid.

Similarly, if we try to apply Rule for Exponents 4 to rational powers we get $(ab)^{\frac{1}{m}} = a^{\frac{1}{m}} b^{\frac{1}{m}}$. For even values of m, this rule only makes sense when both $a^{\frac{1}{m}}$ and $b^{\frac{1}{m}}$ make sense. When m is even, a and b cannot be negative even if their product (ab) is positive.

[3] The *radicand* is the expression under the radical sign

We rewrite the rules for exponents (*carefully*,...did you get that?) into equivalent statements for radicals.

Important Idea 1.2.5.
For all of the following rules, n and m are integers, $m > 0$, and a and b are real numbers. Further, it is required that $a \geq 0$ and $b \geq 0$ when m is even.

1. $(a^{\frac{1}{m}})^n = a^{\frac{n}{m}} = (a^n)^{\frac{1}{m}}$, *so we get:*
   $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ *and* $a^{\frac{n}{m}} = \sqrt[m]{a^n}$ *and so* $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$. where $a \geq 0$ when m even
2. $(ab)^{\frac{1}{m}} = a^{\frac{1}{m}} b^{\frac{1}{m}}$, *and so* $\sqrt[m]{ab} = \sqrt[m]{a}\,\sqrt[m]{b}$. , $a, b \geq 0$ when m is even
3. $\left(\frac{a}{b}\right)^{\frac{1}{m}} = \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}$, *so we get* $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$ *where* $b \neq 0$. $a, b > 0$ when m is even

Example 1.2.10.
Rewrite each of the following into expressions using radicals. Simplify.

1. $9^{\frac{1}{2}} = \sqrt{9} = 3$
2. $(-64)^{\frac{1}{3}} = \sqrt[3]{-64} = -4$
3. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ *or* $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$
4. $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}}$
5. $(\sqrt{7})^{-\frac{1}{3}} = (7^{\frac{1}{2}})^{-\frac{1}{3}} = 7^{-\frac{1}{6}} = \frac{1}{\sqrt[6]{7}}$

Example 1.2.11.
Rewrite each of the following into expressions using rational exponents. Simplify.

$x > 0$

1. $\frac{1}{\sqrt{x}} + \sqrt{x} = x^{-\frac{1}{2}} + x^{\frac{1}{2}} = \frac{x^{1/2}(1+x)}{x}$ Also $\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}$, so $\frac{1}{\sqrt{x}} + \sqrt{x} = \frac{\sqrt{x}}{x} + \sqrt{x} = \frac{\sqrt{x} + x\sqrt{x}}{x}$

$x \geq 0$

2. $\sqrt[3]{\sqrt{x}} = \sqrt[3]{x^{\frac{1}{2}}} = (x^{\frac{1}{2}})^{\frac{1}{3}} = x^{\frac{1}{6}}$

$z, x \geq 0$

3. $\sqrt[5]{\frac{\sqrt{x}\,\sqrt[3]{y}}{\sqrt[4]{z^2}}} = \sqrt[5]{\frac{x^{\frac{1}{2}} y^{\frac{1}{3}}}{z^{\frac{2}{4}}}} = \left(\frac{x^{\frac{1}{2}} y^{\frac{1}{3}}}{z^{\frac{1}{2}}}\right)^{\frac{1}{5}} = \frac{x^{\frac{1}{10}} y^{\frac{1}{15}}}{z^{\frac{1}{10}}}$

Need $z^2 > 0$ b/c 4 is even
$z \in \mathbb{R}$ fulfils this, but
$z^{1/10}$ must be positive, so
ultimately
$\sqrt[4]{z^2}$ requires $z > 0$.

Because rational exponents are radicals, the Rules for Exponents give us rules for working with radicals . We don't want to always have to rewrite radical expressions into exponential expressions before working. We can use the radical forms of the rules in Important Idea 1.2.5 to simplify radical expressions.

This goes back to p. 13, last two ¶.

Example 1.2.12.

1. $\sqrt{6} \cdot \sqrt{7} = \sqrt{42}$
2. $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$
3. $\frac{\sqrt{24}}{\sqrt{18}} = \frac{\sqrt{4 \cdot 6}}{\sqrt{9 \cdot 2}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{9}\sqrt{2}} = \frac{2\sqrt{6}}{3\sqrt{2}} = \frac{2}{3} \cdot \frac{\sqrt{6}}{\sqrt{2}} = \frac{2}{3}\sqrt{\frac{6}{2}} = \frac{2}{3}\sqrt{3}$

4. $\dfrac{\sqrt{24}}{\sqrt{18}} = \sqrt{\dfrac{24}{18}} = \sqrt{\dfrac{4}{3}} = \dfrac{\sqrt{4}}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

In Example 1.2.12 we saw two ways of handling $\frac{\sqrt{24}}{\sqrt{18}}$. The answers that we got do not look identical. However, they are equal. We can use the Rules for Exponents to rewrite fractions where there is a radical expression in the numerator or in the denominator or in both. There are situations where we will not want to have a radical expression in a numerator or not want to have a radical expression in a denominator. The process of eliminating a radical from the numerator or the denominator is called *rationalizing* the numerator or denominator. We rationalize the fraction by multiplying the fraction by an appropriate form of the number 1.

Example 1.2.13.

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2}{3}\sqrt{3} \qquad \textit{Here we chose } 1 = \frac{\sqrt{3}}{\sqrt{3}}$$

We could go the other way: $\dfrac{2}{3}\sqrt{3} = \dfrac{2\sqrt{3}}{3} = \dfrac{2\sqrt{3}}{3}\cdot\dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\cdot 3}{3\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

When rationalizing a numerator or denominator, the choice of "1" depends on the root that you are trying to eliminate. You choose a "1" so that you create a radicand that is a "perfect root."

Example 1.2.14.

$$\frac{6}{\sqrt[3]{2}} = \frac{6}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\underbrace{\sqrt[3]{4}}_{\sqrt[3]{2}\sqrt[3]{2}}} = \frac{6\sqrt[3]{4}}{\sqrt[3]{8}} = \frac{6\sqrt[3]{4}}{2} = 3\sqrt[3]{4} \qquad \textit{Here we chose } 1 = \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$$

The Rules for Exponents tell us how to handle multiplying and dividing radicals. They don't say much about adding or subtracting them.

Important Idea 1.2.6.

You can add or subtract radicals only when they are like terms, i.e., they must ~~have~~ be of the same root and have the same radicand. You add or subtract by appropriately combining their coefficients.

Example 1.2.15.

1. $\sqrt{3} + 5\sqrt{3} = 6\sqrt{3}$
2. $\sqrt{50} - 7\sqrt{2} = \sqrt{25\cdot 2} - 7\sqrt{2} = 5\sqrt{2} - 7\sqrt{2} = -2\sqrt{2}$
3. $4\sqrt{27} - \sqrt{75} = 4\sqrt{9\cdot 3} - \sqrt{25\cdot 3} = 4\cdot 3\sqrt{3} - 5\sqrt{3} = 12\sqrt{3} - 5\sqrt{3} = 7\sqrt{3}$
4. $7\sqrt[3]{3} + 5\sqrt{3} + 8\sqrt[3]{5}$ *cannot be simplified. There are no like terms.*

Comprehension Check 1.2.

1. *Evaluate the following:* $16^{\frac{1}{4}}$ $\quad 27^{-\frac{1}{3}}$ $\quad 25^{\frac{3}{2}}$

   $16^{\frac{1}{4}} = 2$, $27^{-\frac{1}{3}} = \frac{1}{3}$, $= \frac{1}{27^{1/3}}$

   $25^{3/2} = 125$

2. *Simplify:* $\sqrt{18}$ $\quad \dfrac{\sqrt{12}}{\sqrt{8}}$ $\quad \sqrt{75} - \sqrt{50}$

   $3\sqrt{2}$ $\quad \sqrt{3}/\sqrt{2}$ $\quad 5\sqrt{3} - 5\sqrt{2} = 5\sqrt{3} - \sqrt{2}$

3. *Rationalize the denominators:* $\dfrac{6}{\sqrt{3}}$ $\quad \dfrac{1}{\sqrt[5]{27}}$

4. *What is wrong with this statement:* $\sqrt{a^2} = a$ *for all* a *in* $\mathbb{R}$? $a \geq 0$ since $a^{2/2} = (a^{1/2})^2$

   makes sense only for $a \geq 0$

$\sqrt{a^2} = a$ when $a \geq 0$

T or F $\sqrt{a^2} \overset{?}{=} -a$ when $a < 0$ $\quad$ e.g. $\sqrt{(-3)^2} = \sqrt{(-3)(-3)}$

$=$

We have not yet addressed the issue of irrational exponents. What does 2^π mean? We defer that discussion until the chapter on Exponential and Logarithmic Functions. However, it turns out that rational exponents follow the same rules for exponents that we established for integers, so we will just accept that for now.

Example 1.2.16.

$$\frac{2^{\sqrt{5}} \cdot 3^{\pi}}{(2^{-\sqrt{5}} \cdot 3)^{-2}} = \frac{2^{\sqrt{5}} \cdot 3^{\pi}}{2^{2\sqrt{5}} \cdot 3^{-2}} = 2^{\sqrt{5}-2\sqrt{5}} \cdot 3^{\pi--2} = \frac{3^{\pi+2}}{2^{\sqrt{5}}}$$

1.3 Simplifying Expressions

In this section we deal mostly with polynomial expressions. Polynomials are sufficiently important that they get their own chapter later in the text. So, we will not spend time here with formal definition or terminology. However, they are the chosen source for examples of basic algebra rules because they are likely the most familiar to you. The algebra rules also apply to trigonometric expressions, exponential expressions and logarithmic expressions, as we will see in their respective chapters.

1.3.1 Simple Operations

Important Idea 1.3.1.

When adding (or subtracting) algebraic expressions you combine only like terms. You do so by adding (or subtracting) their coefficients.

Example 1.3.1. *Adding expressions*

1. $3x^5 + 7x^5 = 10x^5$
2. $(x-3) + (2x^2 + x) = 2x^2 + 2x - 3$
3. $(2x^5 - 3x^3 + 2x + 3) + (3x^3 + x - 6) = 2x^5 + 3x - 3$
4. $(-5x^4 + 2x^3 - 7) + (4 - x + 3x^2 + x^4 - x^5) = -x^5 - 4x^4 + 2x^3 + 3x^2 - x - 3$
5. $(3x+6) + (5x^2 - 4) + (x^2 + 8x) = 6x^2 + 11x + 2$

When subtracting expressions be sure that you subtract each term in the subtracted expression, not just the leading term.

Example 1.3.2. *Subtracting expressions*

1. $3x^5 - 7x^5 = -4x^5$
2. $(x-3) - (2x^2 + x) = -2x^2 - 3$
3. $(2x^5 - 3x^3 + 2x + 3) - (3x^3 + x - 6) = 2x^5 - 6x^3 + x + 9$
4. $(-5x^4 + 2x^3 - 7) - (4 - x + 3x^2 + x^4 - x^5) = x^5 - 6x^4 + 2x^3 - 3x^2 + x - 11$
5. $(3x+6) - (5x^2 - 4) - (x^2 + 8x) = -6x^2 - 5x + 10$

In Section 1.2 we discussed in detail how to multiply two terms. In brief, the terms do not need to be like terms. You multiply the terms by multiplying the constant coefficients and then multiplying the variable terms by adding the exponents of the common bases. For example: $(8x^5y^2)(2x^7y) = 16x^{12}y^3$
Dividing terms was similar to multiplication except that you divide the coefficients and subtract the exponents of variables with common bases. $(8x^5y^2) \div (2x^7y) = 4x^{-2}y = \frac{4y}{x^2}$.

The fun begins when we mix addition and multiplication. For this we use a rule called the Distributive Property.

1.3.2 The Distributive Property

Important Idea 1.3.2.
The Distributive Property: $a(b+c) = ab + ac$

The Distributive Property tells us how to handle a multiplication problem when one or more of the multipliers contains more than one term. Using the expression in Important Idea 1.3.2 we see that each term in $(b+c)$ gets multiplied by the a. For example, $5(2x-3) = 10x - 15$

This is not a magic formula. It is consistent with what we know about order of operation. $5(2+7) = 10 + 35 = 45$ is consistent with $5(2+7) = 5 \cdot 9 = 45$. The Distributive Property is also consistent with our ideas about addition and multiplication. Suppose your DVD collection consists of four Star-Trek movies, two Monty-Python movies, and a John Wayne thriller. If you were to "triple" your collection (multiply by three) you would then have twelve Star Treks, six Monty-Pythons and three John Wayne movies. You wouldn't just multiply the three times the Star-Trek quantity. $3(4S + 2M + J) = 12S + 6M + 3J$. Notice that we cannot further simplify our movie equation (Captain Kirk, Brave Sir Robin and John Wayne are not like terms).

The expression of the Distributive Property given in Important Idea 1.3.2 is eloquently simple. But the a can represent an expression containing more than one term, and the $(b+c)$ is not restricted to a two-term expression.

Example 1.3.3.
Here we let $a = (2x+3)$ *and* $(b+c) = (4x^3 - x - 7)$

$$\begin{aligned}(2x+3)(4x^3 - x - 7) &= (2x+3)(4x^3) + (2x+3)(-x) + (2x+3)(-7)\\ &= (8x^4 + 12x^3) + (-2x^2 - 3x) + (-14x - 21)\\ &= 8x^4 + 12x^3 - 2x^2 - 17x - 21\end{aligned}$$

Notice that the distributive property actually gets applied four times in this example. First $(2x+3)$ is multiplied by each term in $(4x^3 - x - 7)$. This sets up three more occasions for using the distributive property: we multiply $(4x^3)$ times each term in $(2x+3)$, then we multiply $(-x)$ times each term in $(2x+3)$ and finally we multiply (-7) times each term in $(2x+3)$. After all of the multiplying we had six terms. Each of the two terms in $(2x+3)$ was multiplied times each of the three terms in $(4x^3 - x - 7)$. In practice we don't usually write out all of the steps as done in Example 1.3.3. We simply make sure that each term in the first multiplier gets multiplied times each term in the second multiplier. It does not matter the order in which you do the individual multiplications but you should develop some orderly system so that when dealing with many-term multipliers you don't miss any of the multiplication pairs.

"FOIL"

Some of you may have been taught the "FOIL" method of multiplying expressions containing two terms each. (If you have never heard of "FOIL" just skip this paragraph). "FOIL" is simply a way of helping you to remember the distributive property requirement that you need to multiply each of the two terms in the first expression by each of the two terms in the second expression. To multiply $(a+b)(c+d)$ using "FOIL" you multiply the **F**irst terms (ac), then the **O**utside terms (ad), then the **I**nside terms (bc) and then the **L**ast terms (bd). Of course then you would combine any like terms to finish the problem. This method seems to be effective in getting beginning algebra students to correctly multiply the frequently occurring products of the form $(a+b)(c+d)$. However, students often do not then really understand the distributive property. They then have difficulty performing multiplications when there are more than two terms in one of the expressions. "FOIL" is simply a mnemonic-device (a trick) for remembering all of the multiplication pairs needed to satisfy the distributive property for multiplications of the form $(a+b)(c+d)$. It is only one application of the all-encompassing distributive property. If you are comfortable with "FOIL" by all means use it, but you are now at a level of mathematical sophistication where you need to understand that you are really dealing with the distributive property. (And when called on in class you will sound so much more savvy if you say "apply the distributive property" instead of "FOIL that.")

Example 1.3.4.

1. $(2x+5)(3x-1) = 6x^2 - 2x + 15x - 5 = 6x^2 + 13x - 5$
2. $(a+b+c)(a+b-c) = a^2 + ab - ac + ba + b^2 - bc + ca + cb - c^2$
   $= a^2 + b^2 - c^2 + 2ab$
3. $(\sqrt{x}+\sqrt{y})^2 = (\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y}) = (\sqrt{x})^2 + \sqrt{x}\sqrt{y} + \sqrt{x}\sqrt{y} + (\sqrt{y})^2 = x + \sqrt{xy} + \sqrt{xy} + y =$
   $x + 2\sqrt{xy} + y$
4. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = (\sqrt{x})^2 + \sqrt{x}\sqrt{y} - \sqrt{x}\sqrt{y} - (\sqrt{y})^2 = x - y$

A Quick Look Back at Rationalizing Radical Expressions

Look carefully at Example 1.3.4, parts 3 and 4. In part 3 we took the expression $(\sqrt{x}+\sqrt{y})$ and squared it. The result still contained a radical term. In part 4 we started with the same expression $(\sqrt{x}+\sqrt{y})$, multiplied it by $(\sqrt{x}-\sqrt{y})$ and got a result that did not contain any radicals. This leads us to a way to rationalize fractions that contain sums of square roots.

Expressions in the form $a+b$ and $a-b$ are called *conjugates* of each other. So $(\sqrt{7}+3)$ and $(\sqrt{7}-3)$ are conjugates. When trying to rationalize a numerator or denominator that contains a sum of square roots we use a form of "1" that involves the conjugate. Dealing with sums of more than two terms, or dealing with roots other than squares are a bit more complicated and are not addressed here.

Example 1.3.5.

1. $\dfrac{5}{\sqrt{7}+3} = \dfrac{5}{\sqrt{7}+3}\cdot\dfrac{\sqrt{7}-3}{\sqrt{7}-3} = \dfrac{5(\sqrt{7}-3)}{(\sqrt{7})^2-3^2} = \dfrac{5\sqrt{7}-15}{7-9} = \dfrac{5\sqrt{7}-15}{-2}$

2. $\dfrac{2\sqrt{3}-\sqrt{5}}{\sqrt{2}} = \dfrac{2\sqrt{3}-\sqrt{5}}{\sqrt{2}}\cdot\dfrac{2\sqrt{3}+\sqrt{5}}{2\sqrt{3}+\sqrt{5}} = \dfrac{4\cdot 3-5}{2\sqrt{3}\sqrt{2}+\sqrt{5}\sqrt{2}} = \dfrac{7}{2\sqrt{6}+\sqrt{10}}$

A Very Quick Look Back at Subtraction

When doing subtraction problems you were warned to be sure to subtract each of the terms in the expression that is being subtracted. Consider the following subtraction problem:

$$(4x^3 - 2x^2 + 5x - 7) - (x^3 + x^2 - 2).$$

Another way of thinking of this problem is:

$$(4x^3 - 2x^2 + 5x - 7) + -1(x^3 + x^2 - 2).$$

Certainly these say the same thing...as long as you properly apply the distributive property and multiply each term in the second expression by -1.

Legal Squaring and the "Freshman Mistake"

Suppose we wish to square a simple sum, say $(a+b)$. We need to apply the distributive property:

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

So, it is not true that $(a+b)^2 = a^2 + b^2$

If we wish to cube the sum $(a+b)$ we also use the distributive property:

$$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2 + 2ab + b^2)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

So, it is not true that $(a+b)^3 = a^3 + b^3$.

Important Idea 1.3.3.
$(a+b)^n \neq a^n + b^n$. *There is no such thing as a "Distributive Property of Exponents"*

While the two examples above are fairly straightforward, students so frequently try to apply the false statement $(a+b)^n = a^n + b^n$ that it is sometimes referred to as the "Freshman Mistake."

In fairness, it is an easy mistake to fall victim to because it can appear in subtle forms.

1. We showed in Example 1.3.4, part 3, that $(\sqrt{x} + \sqrt{y})^2 = x + 2\sqrt{xy} + y$, but it is frequently miscalculated as $(\sqrt{x} + \sqrt{y})^2 = x + y$.
2. An even more hidden case of the "Freshman Mistake" occurs when the n value is non-integer. A common sight is: $\sqrt{x^2 + y^2} = x + y$. This is not true (square both sides and see that they don't match). We cannot take $(x^2 + y^2)^{\frac{1}{2}}$ and say that it is equal to $(x^2)^{\frac{1}{2}} + (y^2)^{\frac{1}{2}}$.

Finally, the need to square an expression containing two terms occurs so frequently that it is very helpful to simply learn the pattern of the product and not have to multiply it out each time. $(a+b)^2 = a^2 + 2ab + b^2$. Look at the pattern in the answer. It is the sum of the squares of each of the terms plus twice the product of the terms. So, $(3x+5)^2 = (3x)^2 + 2 \cdot 3x \cdot 5 + 5^2 = 9x^2 + 30x + 25$. Practice so that you can do this quickly.

Comprehension Check 1.3.

1. *Add:* $(3x^2 + 7x) + (2x^2 - x - 5)$
2. *Subtract:* $(3x^2 + 7x) - (2x^2 - x - 5)$
3. *Multiply:* $(3x^2 + 7x)(2x^2 - x - 5)$
4. *Rationalize the denominator:* $\frac{1}{\sqrt{5}+\sqrt{2}}$ – skip for now
5. *How many multiplications must you do to multiply a six-term expression by a four-term expression?* $6 \cdot 4$

1.4 Factoring

In the previous section we took expressions and multiplied them using the distributive property. In this section we are going the other way. We will start with a multiplied expression and try to find two (or more) multipliers that compose it. The multipliers are called *factors*. Finding the multipliers is called factoring. The factoring process contains a certain amount of "trial and error," but this can be minimized by learning a few things to look for and by practicing.

First Strategy – Look for Factors Common to all Terms

Look first to see if there are any expressions (numbers, variables, sums) that are factors (multipliers) in *every* term of the expression you wish to factor. We generally write that factor in front and then figure out, term by term, what the remaining expression (now also a factor) would have to be in order that their product is the original expression.

Example 1.4.1.

$$4x^3 - 6x^2 + 12x = 2x(2x^2 - 3x + 6)$$

In Example 1.4.1 we saw that $2x$ was a common factor in every term of the original expression. So we "pulled out" the $2x$ and then had to figure out, term by term, what was necessary to include in the remaining expression in order that their product was equal to the original $4x^3 - 6x^2 + 12x$. Verify that we indeed found the correct second factor.

Sometimes the factor that is present in each term of the original expression is a sum.

Example 1.4.2.

$$3(x + 2) + x(x + 2) = (x + 2)(3 + x)$$

Sometimes there is further factoring or simplifying to do after the first factorization. In Example 1.4.3 this is true. Also, there is more than one factor common to each term. One of the common factors is a sum.

Example 1.4.3.

$$x^4(x + 1) - x^3(x + 1)^2 = x^3(x + 1)[x - (x + 1)]$$
$$= x^3(x + 1)(-1) = -x^3(x + 1)$$

There is another way to do this problem. We can multiply out the original expression and then "start from scratch" to factor the expression. Almost always this is more work, but the final factorization should be the same.

$$x^4(x+1) - x^3(x+1)^2 = x^5 + x^4 - x^3(x^2+2x+1)$$
$$= x^5 + x^4 - x^5 - 2x^4 - x^3 = -x^4 - x^3 = -x^3(x+1)$$

In Example 1.4.4 we have three problems that involve more "interesting" exponents. Do not let this bother you. The technique is the same. Previously when we found a common base in each term we identified a common factor by choosing that base along with the lowest exponent. We do the same here. Then we use the rules for exponents to decide, term by term, what the remaining factor should be. Look carefully at these examples and verify that they work. Expressions like the last one occur fairly commonly in calculus (something to look forward to).

Example 1.4.4.

1.

$$x^5 - 2x^{-3} = x^{-3}(x^8 - 2)$$

2.

$$2x(x-5)^{-3} - 4x^2(x-5)^{-4}$$
$$= 2x(x-5)^{-4}[(x-5) - 2x]$$
$$= 2x(x-5)^{-4}(-x-5)$$

3.

$$4x^3(2x-1)^{-\frac{1}{2}} - 2x(2x-1)^{\frac{3}{2}}$$
$$= 2x(2x-1)^{-\frac{1}{2}}[2x^2 - (2x-1)^2]$$
$$= 2x(2x-1)^{-\frac{1}{2}}[2x^2 - (4x^2 - 4x + 1)]$$
$$= 2x(2x-1)^{-\frac{1}{2}}(-2x^2 + 4x - 1)$$

Second Strategy – Look for Some Familiar Patterns

There are a few factorings that come up sufficiently frequently that it is beneficial to simply know them.

- Perfect Square–sum: $a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$
- Perfect Square–difference: $a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$
- Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
- Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Notice that in the real number system there is no factoring for the "Sum of Squares." $a^2 + b^2$

You will generally want to factor your original expression completely. Sometimes this requires further factoring after making an initial factoring. You can certainly stop factoring if further factoring causes any coefficients to be non-integers.

Of course, in the factorings above a and b can represent any algebraic expression. In each of the examples below, identify the factoring being used and specify the a and b.

Example 1.4.5.

1. $x^2 - 9 = (x+3)(x-3)$
2. $x^4 - 16 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$
3. $8x^3 - 27 = (2x-3)(4x^2+6x+9)$
4. $5x^2 + 10x + 5 = 5(x^2+2x+1) = 5(x+1)^2$

Factoring Expressions of the Form $ax^2 + bx + c$

We begin with expressions where the coefficient of the x^2 term is 1. Let's look at some specific factors and their corresponding products:

$$(x+3)(x+2) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

$$(x+3)(x-2) = x^2 - 2x + 3x - 6 = x^2 + x - 6$$

$$(x-3)(x+2) = x^2 + 2x - 3x - 6 = x^2 - x - 6$$

$$(x-3)(x-2) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$$

Look at the patterns. The x^2 term in the answers comes from the product of the two x terms in the factors. Thus our factors will each begin with a simple x.

The constant term in the answer comes from the product of the two constant terms in the factors. Thus, if the constant term in the answer is positive then the signs of the constants in the factors must be the same. By the same token if the constant term in the answer is negative then the signs of the constants in the factors must be different.

The x term (the middle term) in the answer comes from adding the constant terms of the factors. So if the signs of the constants have been determined to be the same, the sum of those constants is gotten by just finding the <u>sum</u> of the absolute values of the numbers and attaching the appropriate sign. However, if the signs of the factor constants have been determined to be different then the sum of these numbers is found by finding the <u>difference</u> of the absolute values of these numbers and attaching the sign that corresponds to the number with the larger absolute value.

Example 1.4.6.

Factor: $x^2 + 6x + 5$

The sign of the constant term is positive, so the signs in the factors are the same. The sign of the middle term is positive, so the matching signs in the factors are both positive. So far we have, $(x+\ \ )(x+\ \ )$. *Now we just need two numbers whose <u>product</u> is 5 and whose <u>sum</u> is 6. Those numbers are 1 and 5, so our factoring is* $(x+1)(x+5)$. *Check it.*

Example 1.4.7.

Factor: $x^2 + 5x - 14$

The sign of the constant term is negative, so the signs in the factors are different. The sign of the middle term is positive, so the absolute value of the positive factor constant will be larger than the absolute value of the negative factor constant.

So far we have, $(x + larger)(x - smaller)$. Now we just need two numbers whose product is 14 and whose difference is 5. Those numbers are 7 and 2 so our factoring is $(x+7)(x-2)$. Check it. What would the factoring be if the original expression were: $x^2 - 5x - 14$?

Example 1.4.8.

Factor: $x^2 - 7x + 12$
The sign of the constant term is positive, so the signs in the factors are the same.
The sign of the middle term is negative, so the matching signs in the factors are both negative.
So far we have, $(x- \quad)(x- \quad)$. Now we just need two numbers whose product is 12 and whose sum is 7. Those numbers are 3 and 4, so our factoring is $(x-3)(x-4)$. Check it. What would the factoring be if the original expression were: $x^2 + 7x + 12$?

Now we look at expressions where the coefficient of the x^2 term is not 1. If it is simply -1, or any negative number for that matter, it is a good idea to first factor -1 from the expression and then work from there.

Example 1.4.9.

Factor: $-x^2 + 5x + 24$

$$-x^2 + 5x + 24 = -(x^2 - 5x - 24) = -(x-8)(x+3)$$

When we are dealing with a positive coefficient of the x^2 term that is not 1 we can still look at the constant term to determine whether the signs in the factors will be alike or different. Also, it is still true that that product of the constant terms in the factors will be the constant term of the original expression. It is true that the product of the x terms in the factors will be the x^2 term of the original expression. However, after that the factoring can become complicated. It can involve a certain amount of trial and error as we try to find the right factors in the right combinations to get us the desired middle term.

Example 1.4.10.

Factor: $2x^2 - x - 1$
The sign of the constant term is negative so the signs of the factors will be different.
The only factors of the constant term are 1 and 1. So far we have $(\quad +1)(\quad -1)$. The only factors of $2x^2$ are x and $2x$, so our choices become $(x+1)(2x-1)$ and $(2x+1)(x-1)$. We multiply them out and find that the second one is correct.

Example 1.4.11.

Factor: $5x^2 + 13x - 6$.
The sign of the constant term is negative so the signs of the factors will be different.
The only factors of 5 are 5 and 1, but the possible factors of 6 are (1 and 6) or (2 and 3). This leaves us with eight possible factorings (and no guarantee that any of them will work). We can just try the eight possibilities until we find the solution, or we can attempt to increase our odds of success by looking at the middle term: We need a +13 for a coefficient and to get that we will be subtracting. So we will want our 5 to be multiplied by a positive number of at least three. This leaves us only two choices to check, and $(5x-2)(x+3)$ in fact works.

Factoring by "Grouping"

This method of factoring is probably best explained by example. Essentially you think of dividing the terms of your expression into groups, factor each group separately, and then hope to find a factor common to each group. That common factor then becomes a factor of the original.

Example 1.4.12.

$x^3 + 5x^2 - 5x - 25$	*original expression*
$(x^3 + 5x^2) + (-5x - 25)$	*select a grouping*
$x^2(x+5) + -5(x+5)$	*factor each group*
	identify common factor $(x+5)$
$(x+5)(x^2-5)$	*"pull out" common factor and finish*

Another way of grouping that will work is:

$x^3 + 5x^2 - 5x - 25$	*original expression*
$(x^3 - 5x) + (5x^2 - 25)$	*select a grouping (need to rearrange terms first)*
$x(x^2-5) + 5(x^2-5)$	*factor each group*
	identify common factor (x^2-5)
$(x^2-5)(x+5)$	*"pull out" common factor and finish*

Example 1.4.13.

$5x^3 - 10x^2 + 3x - 6$	*original expression*
$(5x^3 - 10x^2) + (3x - 6)$	*select a grouping*
$5x^2(x-2) + 3(x-2)$	*factor each group*
	identify common factor $(x-2)$
$(x-2)(5x^2+3)$	*"pull out" common factor and finish*

Example 1.4.14.

$8x^5 + 12x^3 - 6x^2 - 9$	*Can you find an*
$(8x^5 + 12x^3) + (-6x^2 - 9)$	*alternate grouping*
$4x^3(2x^2+3) - 3(2x^2+3)$	*and arrive at the*
$(2x^2+3)(4x^3-3)$	*same factorization?*

1.5 Algebra of Rational Expressions – Simplifying R.E.s

In this section we review some of the special rules to keep in mind when dealing with rational (fraction) algebraic expressions. We will do this by comparing these to the arithmetic of fractions, with which you are likely much more practiced. So, what do you know about fractions?

The Denominator of a Fraction is NEVER zero.

Fractions are really a statement of division. Division by zero makes no sense. (Just how many times *does* zero go into six?...and don't say "infinity", unless you are prepared to state that $0 \cdot \infty = 6$).

It is clear that $\frac{6}{0}$ is undefined. Some mathematicians don't even like to see it in print, even when followed by "is not defined." It is the "He who must not be named" of the math world. It should be said that $\frac{0}{0}$ is similarly heinous.

So, we must be careful when dealing with fractions that have variables in the denominator. If given the expression $\frac{3}{x-2}$ it must be recognized that this is only meaningful when $x \neq 2$.

Reducing Fractions

The fraction $\dfrac{36}{48}$ can be reduced. We do it by finding common factors in the numerator and denominator and "canceling" them. Actually, "cancel" is not a mathematical operation. What we are really doing is multiplying the fraction by a convenient form of 1.
$\dfrac{36}{48} = \dfrac{3 \cdot 12}{4 \cdot 12} = \dfrac{3 \cdot 12}{4 \cdot 12} \cdot \dfrac{\frac{1}{12}}{\frac{1}{12}} = \dfrac{3}{4}$. We normally don't bother to write the multiplication step and we call our manipulation "canceling."

We can reduce $\dfrac{3x^2yz^2}{6xy^5z^2}$ to $\dfrac{x}{2y^4}$ using our Rules for exponents but it can easily be thought of as canceling the factors that are common in the numerator and denominator.

Notice that we reduce the fraction only when it is factored. We would not think of doing something like this: $\dfrac{36}{48} = \dfrac{21+15}{33+15} = \dfrac{21}{33}$. Unfortunately sometimes students will reduce $\dfrac{3+x}{y+x}$ to $\dfrac{3}{y}$. This is quite incorrect. We can only reduce a fraction where both the numerator and denominator are factored and we find a common factor.

So, how do we know when an expression is factored? Simply, the expression is written as a multiplication of two or more expressions, called factors. The factors themselves may be sums but the original expression must ultimately be expressed only as a multiplication problem. Another way to think of this is that the only "+" or "−" signs in a factored expression must be included *within* a factor. There are no "+" or "−" signs *between* factors.

Consider $\dfrac{2xy^2}{xy-y}$. The numerator is all multiplication; it is factored. The denominator is a sum; it is not factored. We cannot cancel x's or y's because the ones in the denominator are not factors. We can only cancel common factors. If we want to reduce this fraction we must first factor the denominator. $\dfrac{2xy^2}{xy-y} = \dfrac{2xy^2}{y(x-1)}$ Now we can cancel the common y factor, resulting in the reduced fraction $\dfrac{2xy}{x-1}$.

There is one other point that must be made. When we look at $\dfrac{2xy^2}{xy-y}$ it is clear that $y \neq 0$. However, when we look at $\dfrac{2xy}{x-1}$ this is not so obvious. When we reduced the original fraction we were really multiplying the numerator and denominator by 1 in the form $\frac{\frac{1}{y}}{\frac{1}{y}}$, so we could only do this operation if $y \neq 0$. So, when we reduce a fraction we must also recognize the stipulation under which we are performing the operation. It is not correct to simply say "$\dfrac{2xy^2}{xy-y} = \dfrac{2xy}{x-1}$." We must say "$\dfrac{2xy^2}{xy-y} = \dfrac{2xy}{x-1}$, when $y \neq 0$." This is not optional, it is essential to the meaning of "equal."

Example 1.5.1.

$$\frac{9x^2+9x}{2x+2} = \frac{9x(x+1)}{2(x+1)} = \frac{9x}{2} \quad \textit{when} \quad x \neq -1$$

Example 1.5.2.

$$\frac{x^2+8x-20}{x^2+11x+10} = \frac{(x+10)(x-2)}{(x+10)(x+1)} = \frac{x-2}{x+1} \quad when \quad x \neq -10$$

Notice in Example 1.5.2 that $x \neq -1$ also. However, we don't need to state this explicitly in the final equality statement because $x \neq -1$ is still clear in the expression $\frac{x-2}{x+1}$. This condition for existance did not become invisible like $x \neq -10$ did.

Example 1.5.3.

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{(x-1)\cdot 1} = x^2+x+1 \quad when \quad x \neq 1$$

In Example 1.5.3 the denominator does not look factored. Explain why we are allowed to cancel here. An analogous arithmetic example is: $\frac{21}{7} = \frac{7\cdot 3}{7} = 3$.

X **Comprehension Check 1.4.**

What is the difference between $\frac{x^2-1}{x+1}$ *and* $x-1$*?*

Multiplying and Dividing Fractions

When we multiply fractions, we multiply the numerators together and multiply the denominators together to get our resulting fraction. $\frac{2}{3}\cdot\frac{5}{8} = \frac{2\cdot 5}{3\cdot 8} = \frac{10}{24} = \frac{5}{12}$. We can reduce the fraction after mulitplying or we could do some reducing before multiplying. We still cancel only factors from the numerators with like factors in the denominators. $\frac{2}{3}\cdot\frac{5}{8} = \frac{1}{3}\cdot\frac{5}{4} = \frac{5}{12}$. Usually there is less work if you reduce before multiplying. We handle multiplying rational expressions in exactly the same way that we multiply fractions.

Example 1.5.4.

$$\frac{x+13}{x^4-3x^3}\cdot\frac{x(3-x)}{5} = \frac{(x+13)}{x^3(x-3)}\cdot\frac{-x(x-3)}{5} = \frac{(x+13)(-1)}{x^2\cdot 5}$$

$$= \frac{-x-13}{5x^2} \quad when \quad x \neq 3, 0$$

Example 1.5.5.

$$\frac{4y-16}{5y+15}\cdot\frac{2y+6}{y-4} = \frac{4(y-4)}{5(y+3)}\cdot\frac{2(y+3)}{(y-4)} = \frac{4\cdot 2}{5} = \frac{8}{5} \quad when \quad y \neq 4 \quad and \quad y \neq -3$$

The operation of division is really just multiplication by the reciprocal of the divisor. $5 \div 4$ is really $5\cdot\frac{1}{4} = \frac{5}{4}$. Indeed fractional notation is simply another way of writing a division problem. $\frac{2}{3}\div\frac{1}{5}$ is the same as $\frac{\frac{2}{3}}{\frac{1}{5}}$. So, to simplify this complex fraction, this division problem, we simply multiply $\frac{2}{3}$ by the reciprocal of its divisor. The whole problem looks like: $\frac{2}{3}\div\frac{1}{5} = \frac{2}{3}\cdot\frac{5}{1} = \frac{10}{3}$. We summarize this with the following Important Idea.

Important Idea 1.5.1.

The following expressions are equivalent: $\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$.

We use the same rule for dividing rational expressions. We simply turn them into multiplication problems by using the reciprocal of the divisor.

Example 1.5.6.

$$\frac{x^2-14x+49}{x^2-49} \div \frac{3x-2}{x+7} = \frac{x^2-14x+49}{x^2-49} \cdot \frac{x+7}{3x-2} = \frac{(x-7)^2}{(x+7)(x-7)} \cdot \frac{(x+7)}{3x-2}$$

$$= \frac{x-7}{3x-2} \quad \textit{when } x \neq 7 \textit{ and } x \neq -7$$

Adding and Subtracting Fractions

When we add or subtract fractions we need to have a common denominator. This is because we can only add or subtract terms that are alike, and fractions with the same denominator represent entities with the same unit of measure. As you look at the next two arithmetic examples, think about the thought process you use to find the common denominator and perform the addition and subtraction.

Example 1.5.7.

$$\frac{3}{10} + \frac{7}{20} = \frac{6}{20} + \frac{7}{20} = \frac{13}{20}$$

$$\frac{5}{12} - \frac{11}{15} = \frac{25}{60} - \frac{44}{60} = -\frac{19}{60}$$

In Example 1.5.7 we decide on a common denominator by finding some number that is a multiple of both of the denominators involved. 20 is a multiple of both 10 and 20. 60 is a multiple of both 12 and 15. It is helpful if we can find the smallest multiple when looking for a common denominator, but it isn't necessary. After deciding on a common denominator we rewrite each of the fractions into an equivalent fraction that has the common denominator as its denominator. This is the opposite of reducing a fraction. Once both fractions are rewritten to have the same denominator we can add or subtract them by adding or subtracting their numerators. Sometimes we will be able to reduce the final answer. In Example 1.5.7 the answers could not be further simplified.

We use this same process for adding and subtracting rational expressions.

Example 1.5.8.

$$\frac{2x-1}{x+3} - \frac{1-x}{x+3} = \frac{(2x-1)-(1-x)}{x+3} = \frac{2x-1-1+x}{x+3} = \frac{3x-2}{x+3}$$

Example 1.5.9.

$$\frac{2x}{x^2-x-2} + \frac{10}{x^2+2x-8} = \frac{2x}{(x-2)(x+1)} + \frac{10}{(x+4)(x-2)}$$

$$= \frac{2x(x+4)}{(x-2)(x+1)(x+4)} + \frac{10(x+1)}{(x+4)(x-2)(x+1)}$$

$$= \frac{2x^2 + 8x + 10x + 10}{(x+4)(x-2)(x+1)} = \frac{2x^2 + 18x + 10}{(x+4)(x-2)(x+1)}$$

Example 1.5.10.

$$\frac{2}{x+1} + \frac{2}{x-1} - \frac{1}{x^2-1} = \frac{2}{(x+1)} + \frac{2}{(x-1)} - \frac{1}{(x+1)(x-1)}$$
$$= \frac{2(x-1)}{(x+1)(x-1)} + \frac{2(x+1)}{(x+1)(x-1)} - \frac{1}{(x+1)(x-1)}$$
$$= \frac{2x - 2 + 2x + 2 - 1}{(x+1)(x-1)} = \frac{4x-1}{(x+1)(x-1)}$$

1.6 Solving Equations

The ability to factor an expression can be quite useful in solving equations. If we can take an equation and rewrite it so that one side of the equation is zero and the other side of the equation is completely factored, then we can use the following fact to help us solve the equation.

Important Idea 1.6.1. *A Property of Zero*

If $a \cdot b = 0$ *then* $a = 0$ *or* $b = 0$.

This idea might seem obvious, but don't overlook its power and helpfulness. It says that if you have two numbers, or two algebraic expressions, a and b whose product is zero, then at least one of them must be zero. It could be that both a and b are zero. This property does not work for any other number but zero. There is no corresponding "Property of Sixes." Indeed, can you think of two numbers, a and b where $a \cdot b = 6$ but neither a nor b is 6? (If not then you should probably take a quick break, maybe get a snack or legal beverage, and come back shortly). Let's see how this Property of Zero helps us to solve equations.

Suppose we want to solve the equation $x^2 + 2x - 8 = 7$ for x.

$x^2 + 2x - 8 = 7$	original equation
$x^2 + 2x - 15 = 0$	subract 7 from both sides so that one side equals zero
$(x+5)(x-3) = 0$	rewrite the non-zero side into factored form
$x + 5 = 0$ or $x - 3 = 0$	apply the Property of Zeros
$x = -5$ or $x = 3$	solve for x

Comprehension Check 1.5.

What is wrong with the following work?

$x^2 + 2x - 8 = 7$

$(x+4)(x-2) = 7$

$x + 4 = 7$ *or* $x - 2 = 7$

$x = 3$ *or* $x = 9$

Example 1.6.1.

Solve for x.

$$(x^2 + 2x)(x - 5) = 0$$

$$x(x+2)(x-5)=0$$
$$x=0 \text{ or } x+2=0 \text{ or } x-5=0$$
$$x=0 \text{ or } x=-2 \text{ or } x=5$$

Example 1.6.2.
Solve for x.

$$x^5+x^2=-2x^3-2$$
$$x^5+x^2+2x^3+2=0$$
$$(x^5+2x^3)+(x^2+2)=0$$
$$x^3(x^2+2)+(x^2+2)=0$$
$$(x^2+2)(x^3+1)=0$$
$$x^2+2=0 \text{ or } x^3+1=0$$
$$x=-1 \text{ is the only solution}$$

Important Idea 1.6.2.
If a fraction is equal to zero, then its numerator is equal to zero. In other words:

$$\text{If } \frac{a}{b}=0, \text{ then } a=0.$$

Example 1.6.3.
Solve for x.

$$\frac{2}{x+1}+\frac{2}{x-1}=\frac{1}{x^2-1}$$
$$\frac{2}{(x+1)}+\frac{2}{(x-1)}-\frac{1}{(x^2-1)}=0$$

Using the result from Example 1.5.10 this becomes:

$$\frac{4x-1}{(x+1)(x-1)}=0$$

$$4x-1=0 \qquad so, \qquad x=\frac{1}{4}.$$

In solving some equations it can be useful to square both sides of the equation. This is valid but we must be careful to check our solutions in the original problem. The equation $x=\sqrt{25}$ does not have the same solution set as the equation $x^2=25$. So, squaring both sides of an equation will get you all of the solutions to the original problem, but as a bonus could give you some solutions that are not valid in the original.

Important Idea 1.6.3.
When squaring both sides of an equation, make sure that you:

- *square the* entire *side on both sides. Do not fall victim to the "Freshman Mistake."*
- *check each solution in the original equation so that you eliminate invalid solutions.*

Example 1.6.4.
Solve for x.

$$\begin{aligned}\sqrt{2x+3} &= 5\\ (\sqrt{2x+3})^2 &= 5^2\\ 2x+3 &= 25\\ 2x &= 22\\ x &= 11\end{aligned}$$

Check: $\sqrt{2\cdot 11+3} = \sqrt{22+3} = \sqrt{25} = 5 \quad \checkmark$

Example 1.6.5.
Solve for x.

$$\begin{aligned}x+\sqrt{x-4} &= 4\\ \sqrt{x-4} &= 4-x\\ (\sqrt{x-4})^2 &= (4-x)^2\\ x-4 &= 16-8x+x^2\\ 0 &= x^2-9x+20\\ 0 &= (x-5)(x-4)\end{aligned}$$

$$x=5 \quad or \quad x=4$$

Check:
$5+\sqrt{5-4} = 5+1 = 6 \neq 4$, *so REJECT this solution.*
$4+\sqrt{4-4} = 4-0 = 4 \quad \checkmark$
So, $x=4$ is the only solution.

In the previous example it was useful to move the x term to the other side of the equation. Do you see why? If not, try to solve the problem without moving the x first. Be sure to square the *entire* left side.

What happens if you have two or more radicals in your original equation? Then you are stuck, but you should still try to make the squaring as easy as possible by having as few complex radicals as possible on the same side of the equation.

Example 1.6.6.
Solve for x.

$$\begin{aligned}\sqrt{3x-2}+\sqrt{3x+1} &= 3\\ \sqrt{3x-2} &= 3-\sqrt{3x+1}\\ (\sqrt{3x-2})^2 &= (3-\sqrt{3x+1})^2\\ 3x-2 &= 9-6\sqrt{3x+1}+(3x+1)\\ 3x-2-9-(3x+1) &= -6\sqrt{3x+1}\\ -12 &= -6\sqrt{3x+1}\\ 2 &= \sqrt{3x+1}\\ 4 &= 3x+1\\ 3 &= 3x\\ x &= 1\end{aligned}$$

Check:
$\sqrt{3\cdot 1-2}+\sqrt{3\cdot 1+1} = \sqrt{3-2}+\sqrt{3+1} = \sqrt{1}+\sqrt{4} = 1+2 = 3 \quad \checkmark$

1.7 Exercises

Problems for Section 1.1

Problem 1. If you still have not read "Dear Student" at the beginning of the text, do so.

Problem 2. Write a decimal expression for each of the following rational numbers:

$$\text{(a) } \frac{3}{1} \quad \text{(b) } \frac{3}{2} \quad \text{(c) } \frac{3}{3} \quad \text{(d) } \frac{3}{4} \quad \text{(e) } \frac{3}{5} \quad \text{(f) } \frac{3}{6} \quad \text{(g) } \frac{3}{7}.$$

Problem 3. Show that the following are rational numbers by expressing them in the form $\frac{a}{b}$. You do not have to reduce your fraction. An example is given below for changing repeating decimals into fractions. You will have to make appropriate modifications to the example to fit it to some more "complex" problems.

$$\text{(a) } 12 \quad \text{(b) } -.36 \quad \text{(c) } 21.7 \quad \text{(d) } -.\overline{5} \quad \text{(e) } .\overline{27} \quad \text{(f) } 1.\overline{27} \quad \text{(g) } .1\overline{27}$$

Example: Change $.\overline{4}$ to rational form.
Let $n = .\overline{4}$. Then $n = .444\ldots$, and $10n = 4.444\ldots$.
If we subtract $10n - n$ we get $9n = (4.444\ldots) - (.444\ldots)$.
The repeated fours cancel and we are left with $9n = 4$, so $n = \frac{4}{9}$.

Problem 4. Given the expression $3x + 5x^2y - \frac{x}{2} + y^3 + 5y^2x - 2y^3 + 7 + x^2 - yx^2$

1. Find all sets of like terms.
2. Identify the coefficient for each term.

Problem 5. Represent each of the following sets (1) algebraically, (2) in interval notation, and (3) on a number line.

1. "x is a number less than 7 or greater than or equal to 10"
2. "y is a number greater than π and less than or equal to 5"
3. "z is a negative number greater than -4.

Problem 6. Describe each of the sets in the previous problem as open or closed and as bounded or unbounded.

Problem 7. Identify each of the numbers below as integer, rational, and/or irrational. Then draw a single number line and place the numbers on it.

$$\text{(a) } -\frac{1}{2} \quad \text{(b) } 0 \quad \text{(c) } -2\pi \quad \text{(d) } \sqrt{3} \quad \text{(e) } 1 \quad \text{(f) } \frac{\sqrt{2}}{2} \quad \text{(g) } \frac{7}{3} \quad \text{(h) } (1-\sqrt{2})$$

Problems for Section 1.2

Problem 1. Evaluate the following:

$$\text{(a) } 32^{\frac{4}{5}} \quad \text{(b) } 17^0 \quad \text{(c) } 8^{-\frac{1}{3}} \quad \text{(d) } 4^{\frac{3}{2}} \quad \text{(e) } 100^{\frac{1}{2}} - 64^{\frac{1}{2}} \quad \text{(f) } (100-64)^{\frac{1}{2}}$$

Problem 2. Rewrite the following expressions so that they only contain positive exponents. Simplify.

$$\text{(a)}\ \frac{16x^2yz^{-1}}{2xy^{-3}z} \qquad \text{(b)}\ \frac{5x^{-2}yz^3}{(2x)^{-2}(yz^3)^2} \qquad \text{(c)}\ \frac{(4x)^{\frac{1}{2}}y}{8x^{-\frac{1}{2}}y^3} \qquad \text{(d)}\ \left(\frac{xy^5}{3x^{-2}yz}\right)^{-2} \qquad \text{(e)}\ \left(\frac{a^{-2}}{b^{-2}}+\frac{b^{-2}}{a^{-1}}\right)^{-1}$$

Problem 3. Change the following to exponential form (eliminate the radical sign):

$$\text{(a)}\ \sqrt[3]{x^5} \qquad \text{(b)}\ (\sqrt[5]{2x})^3 \qquad \text{(c)}\ \left(\sqrt{\frac{x}{y^3}}\right)^5 \qquad \text{(d)}\ \frac{x}{\sqrt[5]{x^3}} \qquad \text{(e)}\ \sqrt[6]{\sqrt[3]{x^4}}$$

Problem 4. Change the following to radical form:

$$\text{(a)}\ x^{\frac{1}{3}} \qquad \text{(b)}\ -x^{\frac{1}{2}} \qquad \text{(c)}\ x^{\frac{9}{5}} \qquad \text{(d)}\ -3x^{\frac{2}{3}} \qquad \text{(e)}\ 2(xy)^{-\frac{3}{4}}$$

Problem 5. For what values of x are each of the following defined?

$$\text{(a)}\ \sqrt{x} \qquad \text{(b)}\ \sqrt{-x} \qquad \text{(c)}\ \sqrt{x^2} \qquad \text{(d)}\ \frac{1}{\sqrt{x}} \qquad \text{(e)}\ \sqrt{x-6} \qquad \text{(f)}\ \sqrt[3]{x}$$

Problem 6. Which, if any, of the following are equal to $\sqrt{(-x)^5}$?

$$\text{(a)}\ -x^{\frac{5}{2}} \qquad \text{(b)}\ (-x)^{\frac{5}{2}} \qquad \text{(c)}\ -x^{\frac{2}{5}} \qquad \text{(d)}\ (-x)^{\frac{2}{5}} \qquad \text{(e)}\ x^{\frac{-2}{5}} \qquad \text{(f)}\ \sqrt{-x^5} \qquad \text{(g)}\ (\sqrt{-x})^5$$

Problem 7. Rationalize the denominators for the following fractions:

$$\text{(a)}\ \frac{1}{\sqrt{2}} \qquad \text{(b)}\ \frac{1}{\sqrt{3}} \qquad \text{(c)}\ \frac{1}{\sqrt{5}} \qquad \text{(d)}\ \frac{1}{\sqrt{a}}$$

You might have figured out that this is a good one to know readily.

Problem 8. Rationalize the denominators for the following fractions:

$$\text{(a)}\ \frac{3}{\sqrt[3]{5}} \qquad \text{(b)}\ \frac{1}{\sqrt[3]{9}} \qquad \text{(c)}\ \frac{\sqrt{7}}{\sqrt[3]{49}} \qquad \text{(d)}\ \frac{6}{\sqrt{5}+\sqrt{2}} \qquad \text{(e)}\ \frac{\sqrt{3}}{\sqrt{3}-\sqrt{7}} \qquad \text{(f)}\ \frac{\sqrt{2}}{\sqrt{6}-2}$$

Problem 9. Simplify the following expressions as much as possible.

$$\text{(a)}\ \sqrt{8}\cdot\sqrt{2} \qquad \text{(b)}\ \sqrt[3]{6}\cdot\sqrt[3]{9} \qquad \text{(c)}\ \frac{4}{\sqrt[3]{16}}$$

$$\text{(d)}\ \sqrt{8}+\sqrt{2} \qquad \text{(e)}\ \sqrt{18}-\sqrt{14}+\sqrt{32} \qquad \text{(f)}\ \sqrt{75}+\sqrt{48}+\sqrt{12}-\sqrt{50}$$

$$\text{(g)}\ \left(\frac{\sqrt{3}-\sqrt{6}}{2}\right)^2$$

Problem 10. Where is the first error in the following "proof" that $2=-2$?

$$2=\sqrt{4}=\sqrt{(-2)\cdot(-2)}=\sqrt{-2}\cdot\sqrt{-2}=-2.$$

Problems for Section 1.3

Problem 1. Add or subtract and simplify:

(a) $(6x^3 + 2x - 5) + (3x^3 + 2x)$ (b) $(3x^2y + 2y^2x) + (5xy^2 - 7yx^2)$
(c) $(3x^3 + 2x) - (6x^3 + 2x - 5)$ (d) $3x^2 - [4x - x(2x + 1)]$

Problem 2. Multiply and simplify:

(a) $(x + 2)^2$ (b) $(5 - 8x)^2$ (c) $2x(4x^3 - x - 1)$
(d) $(2x + 3)(4x^3 - x - 1)$ (e) $(x + 2)(2x - 1)(x - 2)$ (f) $(x^2 - x + 5)(3x^2 + x - 5)$
(g) $2x^{\frac{1}{3}}(3x^{\frac{2}{3}} + x^3)$

Problem 3. Find numbers a and b, and c and d that illustrate the following:

(a) $(a + b)^2 \neq a^2 + b^2$ (b) $\sqrt{c^2 + d^2} \neq c + d$

Problems for Section 1.4

Problem 1. Evaluate without a calculator:

(a) $4,444,444^2 - 4,444,443^2$ (b) $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

Problem 2. Factor completely:

(a) $x^2 - x - 12$ (b) $x^2 - 13x + 22$ (c) $x^2 + 2x - 35$
(d) $3x^2 - 18x + 15$ (e) $4x^2 + 40x + 100$ (f) $2x^2 - x^{-1} + x^{-4}$
(g) $x^3 - 2x^2 + 4x - 8$ (h) $3x^2 + 5x + 2$ (i) $3x^3 + 24$
(j) $-2x^2 + 6x + 8$ (k) $1 - x^{16}$ (l) $x^5 + 2x^3 + x^2 + 2$
(m) $6x(2x + 1)^{-\frac{1}{2}} + 3x^2(2x + 1)^{\frac{1}{2}}$ (n) $x^4 + 4x^2 - 5$ (o) $2x^5 - 16x^4 + 32x^3$
(p) $12x^2 + 16x - 3$ (q) $-27x^3 + 1$ (r) $4x^3 - 12x^2 - 9x + 27$
(s) $(x + 2)^2 - 9$

Problems for Section 1.5

Problem 1. Simplify (reduce) the following rational expressions. Be sure to indicate any restrictions that must be made on the variables in order that your simplified answer is equivalent to the original.

(a) $\dfrac{3x^3y + 12xy}{6xy^2}$ (b) $\dfrac{x^3 + y^3}{x^2 - y^2}$ (c) $\dfrac{2x^2 + x - 1}{2x^2 + 5x - 3}$ (d) $\dfrac{x - 2(x + 3) + 3}{x + 3}$

Problem 2. For what values of x is it NOT true that $\dfrac{(x + 2)(x - 3)x}{(x - 4)(x + 2)x} = \dfrac{x - 3}{x - 4}$?

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Problem 3. Perform the indicated operation(s) and simplify. Be sure to indicate any restrictions that must be made on x in order that your simplified answer is equivalent to the original.

(a) $\dfrac{3x+2}{x-2}+\dfrac{x+1}{4-2x}$ (b) $x-\dfrac{2}{x}+1$ (c) $\dfrac{x^2}{x^2+2x+1}+\dfrac{1}{3x+3}+\dfrac{1}{3}$

(d) $\dfrac{2x+1}{x+2}-\dfrac{x-2}{x+3}$ (e) $1-\dfrac{x-y}{y-x}$ (f) $\dfrac{1+\frac{3}{x}}{x-\frac{9}{x}}$

$(x+9)(x+4)$

(g) $\dfrac{9-x^2}{x^2+5x+6}\cdot\dfrac{x+2}{x-3}$ (h) $\dfrac{x^2-16}{2x^2+10x+8}\div\dfrac{x^2+13x+36}{x^3+1}$ (i) $\dfrac{2x^2+7x+3}{4x^2-1}\div(x+3)$

$2(x+4)(x+1)$

(j) $\dfrac{\left(\dfrac{x^3}{2x^2+3x}\right)}{\left(\dfrac{4x^2-6x}{12x+18}\right)}$ (k) $(4x^2+7x+3)\div(x^2+5x+4)$

Problems for Section 1.6

Problem 1. Solve the following equations for x.

(a) $x^3-4x=0$ (b) $x^3+4x=0$ (c) $-2x^2-15x+27=0$

(d) $x^3-2x^2=3x$ (e) $x(x+2)=99$ (f) $\dfrac{x^2-4x}{x+3}=\dfrac{5}{x+3}$

(g) $x(3x-23)=8$ (h) $\dfrac{x}{4}(x+1)=3$ (i) $\dfrac{3}{x+5}+\dfrac{4}{x}=2$

(j) $\sqrt{x+7}=x-13$ (k) $\sqrt{5x+9}-x=-1$ (l) $\sqrt{3x-2}=2+\sqrt{x}$

(m) $\sqrt{3x+6}-\sqrt{x+4}=\sqrt{2}$

1.8 Answers to Exercises

Answers for Section 1.1 Exercises

Answer to Problem 1.
N/A

Answer to Problem 2.

(a) 3 (b) 1.5 (c) 1 (d) .75 (e) .6 (f) .5 (g) $.\overline{428571}$

Answer to Problem 3.

(a) $\dfrac{12}{1}$ (b) $\dfrac{-36}{100}$ (c) $\dfrac{217}{10}$ (d) $\dfrac{-5}{9}$ (e) $\dfrac{27}{99}$ (f) $\dfrac{126}{99}$ (g) $\dfrac{126}{990}$

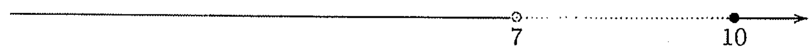
Answer to Problem 4.

Like terms: $5x^2y$ and $-yx^2$; $3x$ and $-\frac{1}{2}x$; y^3 and $-2y^3$

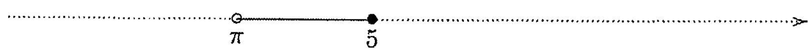
Coefficients, in order: 3, 5, $-\frac{1}{2}$, 1, 5, -2, 7, 1 , -1

Answer to Problem 5.

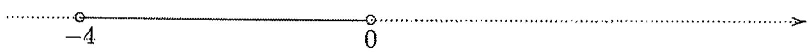
(a) $x < 7$ or $x \geq 10$ $(-\infty, 7) \cup [10, \infty)$



(b) $\pi < y \leq 5$ $(\pi, 5]$



(c) $-4 < z < 0$ $(-4, 0)$



Answer to Problem 6.

(a) half-open, unbounded (b) half-open, bounded (c) open, bounded

Answer to Problem 7.

(a) rational	(b) integer, rational	(c) irrational	(d) irrational
(e) integer, rational	(f) irrational	(g) rational	(h) irrational

Number line values from left to right: -2π, $-\frac{1}{2}$, $(1-\sqrt{2})$, 0, $\frac{\sqrt{2}}{2}$, 1, $\sqrt{3}$, $\frac{7}{3}$.

Answers for Section 1.2 Exercises

Answer to Problem 1.

(a) 16 (b) 1 (c) $\dfrac{1}{2}$ (d) 8 (e) 2 (f) 6

Answer to Problem 2.

(a) $\dfrac{8xy^4}{z^2}$ (b) $\dfrac{20}{yz^3}$ (c) $\dfrac{x}{4y^2}$ (d) $\dfrac{9z^2}{x^6y^8}$ (e) $\dfrac{a^2b^2}{a^3+b^4}$

Answer to Problem 3.

(a) $x^{\frac{5}{3}}$ (b) $(2x)^{\frac{3}{5}}$ (c) $x^{\frac{5}{3}}y^{-\frac{15}{2}}$ (d) $x^{\frac{2}{5}}$ (e) $x^{\frac{2}{9}}$

Answer to Problem 4.

(a) $\sqrt[3]{x}$ (b) $-\sqrt{x}$ (c) $\sqrt[5]{x^9}$ *or* $(\sqrt[5]{x})^9$

(d) $-3\sqrt[3]{x^2}$ *or* $-3(\sqrt[3]{x})^2$ (e) $\frac{2}{(\sqrt[4]{xy})^3}$ *or* $\frac{2}{\sqrt[4]{(xy)^3}}$