

## §2.8 Related Rates

## Steps to Solve a Related Rate

- i) Draw two pictures: one **generic** (arbitrary time  $t$ ) and one **snapshot** (at the particular time of interest, say  $t_0$ ).
- ii) Identify and name quantities of interest in both pictures. In the generic picture, many of these will be functions of time. In the snapshot, some quantities and rates are known.
- iii) Decide what rate you are looking for.
- iv) Determine how quantities are related, especially the quantity whose rate you seek and the quantity whose rate is known.
- v) Differentiate both sides of the equation relating the quantities of interest (be careful to use the chain rule as needed).
- vi) Specialize to the specific time of interest, substituting known values and solve for the rate you seek.

If  $V$  is the volume of a cube with edge lengths  $x$  and the cube expands as time passes, find  $\frac{dV}{dt}$  in terms of  $\frac{dx}{dt}$ .

- a) What is  $\frac{dV}{dx}$  when  $x = 4$  inches and is growing at a rate of 2 inches per minute?
- b) What is  $x$  if the volume is shrinking at 3 cubic inches per minute and the side length is shrinking at 4 inches per minute?
- c) Can a cube have a shrinking volume and a growing sides?

A spherical weather balloon is being inflated at a rate of  $0.5 \text{ m}^3/\text{sec}$ .

- a) How fast is the diameter increasing at the instant the diameter is 2 meters?
- b) How fast is the volume changing at that same instant?
- c) How fast is the surface area changing at that same instant?

As gravel is poured into a conical pile, its volume  $V$  changes with time. As a result, the height  $h$  and radius  $r$  also change with time. Knowing that at any moment  $V = \frac{1}{3}\pi r^2 h$ , the relationship between the changes with respect to time in the volume, radius and height is

$$\text{a) } \frac{dV}{dt} = \frac{1}{3}\pi \left( 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$$

$$\text{b) } \frac{dV}{dt} = \frac{1}{3}\pi \left( 2r \frac{dr}{dt} \cdot \frac{dh}{dt} \right)$$

$$\text{c) } \frac{dV}{dr} = \frac{1}{3}\pi \left( 2rh + r^2 \frac{dh}{dt} \right)$$

$$\text{d) } \frac{dV}{dh} = \frac{1}{3}\pi \left( (r^2)(1) + 2r \frac{dr}{dh} h \right)$$

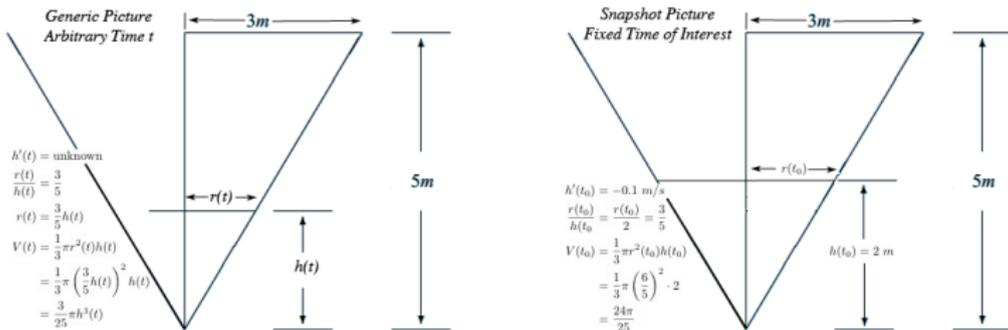
Imagine the following magic triangle. Its base is on a horizontal surface and no matter what you do to its height, the triangle always has area  $10 \text{ cm}^2$ .

If you push down on the top of the triangle so that it becomes shorter at a rate of  $3 \text{ cm/sec}$ , how fast will the length of the base be changing when the triangle is  $5 \text{ cm}$  tall?

My neighbors have a very loud stereo. The volume knob turns half a circle (angles from  $0^\circ$  to  $180^\circ$ ) and the volume of the music is given by the function  $V(\theta) = 110 \sin(\theta/2)$  decibels (dB).

One night at 3:30 in the morning I notice an increase from a volume of 88 dB at a rate of 1 decibel per second! At what rate can I deduce that my neighbor is turning his volume knob?

Water is leaking out of a tank shaped like a right circular cone with height 5 m and top radius 3 m. When the water level in the cone is 2 m, the water level is decreasing at a rate of  $0.1 \frac{m}{s}$ . How fast is the water leaking out of the cone at that moment? (Let's call that moment  $t_0$ .)



Differentiate both sides of  $V(t) = \frac{3}{25}\pi h^3(t)$  and then specialize to time  $t_0$ .

- a) A streetlight hangs 5 meters above the ground. Regina, who is 1.5 meters tall, walks away from the point under the light at a rate of 2 meters per second. How fast is her shadow lengthening when she is 7 meters away from the point under the light?  
(Hint: Use similar triangles.)
- b) Suppose Regina has the ability to magically shrink herself. At what rate must she do this to keep her shadow a constant length of 3 meters? Write this as a function of only her distance from the point under the light.

A revolving beacon from a light house shines on the straight shore, and the closest point on the shore is a pier one half mile from the lighthouse. Let  $\theta$  denote the angle between the lighthouse, pier, and point on the shore where the light shines.

- a) Write the distance from the pier to the point of light as a function of  $\theta$ .
- b) What is the rate of change of the distance from the pier to the point of light with respect to  $\theta$ .
- c) Suppose  $\theta$  is a function of time  $t$ . Give an expression for the rate of change of distance with respect to time  $t$ .
- d) Suppose that the light makes 1 revolution per minute. How fast is the light traveling along the straight beach at the instant it passes over a shorepoint 1 mile away from the shorepoint nearest the searchlight?

Given that a spherical raindrop evaporates at a rate proportional to its surface area, how fast does the radius shrink?

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The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

- a) Find the shortest line segment with endpoints on the  $x$  and  $y$  axes going through the point  $(1, 8)$ .
- b) What is the area of the triangle formed by the shortest line segment?
- c) What is the rate of change of area with respect to the  $x$ -coordinate of the point on the  $x$ -axis?
- d) For which  $x$  is the area increasing?

The speed limit on a stretch of highway is 55 mph. Highway patrol officer, Sgt. Miguel, stations himself at a point, out of view of the motorists, 50 feet off the highway. Miguel is equipped with a radar gun which measures the speed at which a car approaches **his position**.

He takes a reading of suspected speeders by pointing his radar gun at a point on the highway 120 feet from the point on the highway closest to him. The radar gun picks up a reading of 48 feet/sec for a green Chevy driven by Alyssa. How fast is she traveling? Is Alyssa speeding?

At a certain moment, ship  $A$  is 6 miles south and 8 miles west of ship  $B$ . Ship  $A$  at that moment is steaming east at 12 mph, while ship  $B$  is steaming north at 15 mph.

At that moment are the ships approaching each other or separating from each other? At what rate?

Particle  $A$  moves along the positive horizontal axis, and particle  $B$  along the graph of

$$f(x) = -\sqrt{3}x, \quad x \leq 0.$$

At a certain time,  $A$  is at the point  $(5, 0)$  and moving with speed 3 units/sec; and  $B$  is at a distance of 3 units from the origin moving with speed 4 units/sec. At that instant, what rate is the distance between  $A$  and  $B$  changing?