Today's plan:

- Section 1.2.7 : Rankings
- Section 1.3 : One person - Multiple votes; Two alternatives

Section 1.2.7 : Rankings

- Elections are sometimes held for multiple offices at once, with each candidate interested in any position.

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- For example, the Math Club could be electing president, vice-president and treasurer all at once.

Now it's important to get not only the winner, but a ranking of the candidates.

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If we do the **Borda count** method or **pairwise comparisons** method, a ranking falls right out: most points to fewest points.

Example

- Math Club electing president, vice-president and treasurer.

Example

- Math Club electing president, vice-president and treasurer.
- Four candidates **A**, **B**, **C**, and **D**.

Example

They'll be ranked and

- first place is president

Example

They'll be ranked and

- first place is president
- second place is vice-president

Example

They'll be ranked and

- first place is president
- second place is vice-president
- third place is treasurer

a) Get the ranking of candidates using the Borda count method.

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b) Get the ranking of candidates using the pairwise comparisons method.

Borda method:

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We already got:

- **A** gets 45 points
- **B** gets 57 points
- **C** gets 58 points
- **D** gets 40 points

So, ranking with the Borda count method we get:

President:	**C**
Vice-president:	**B**
Treasurer:	**A**

Pairwise comparison method:

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We already got:

- **A** gets 0 points
- **B** gets 3 points
- **C** gets 2 points
- **D** gets 1 points

So, ranking with the Pairwise comparison method we get:

President:	**B**
Vice-president:	**C**
Treasurer:	**D**

Remark

The two methods produce completely different results.

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- the candidate eliminated in the first round in last place
- the candidate eliminated in the second round in second to last place
- and so on...

Remarks:

- In order to get a complete ranking we can't just stop when someone has a majority.

Remarks:

- In order to get a complete ranking we can't just stop when someone has a majority.
- We have to keep going until there are only two candidates.

Example

Get the ranking of the candidates in the Math Club election, using Plurality with Elimination.

Solution

We already got:

- *candidate* **D** *is eliminated in the first round*
- *candidate* **B** *is eliminated in the second round*
- *candidate* **A** *is eliminated in the third round*
- *candidate* **C** *is the winner*

Solution

Therefore, according to the plurality with elimination method we get:

President:	**C**
Vice-president:	**A**
Treasurer:	**B**

(Note: We didn't talk about rankings using the Plurality Method, but it's clear how to do it.)

Section 1.3 : One Person – Multiple Votes; Two Alternatives

Example (Motivating)

At a shareholders assembly, the owners of a company vote on some propositions:

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Example (Motivating)

At a shareholders assembly, the owners of a company vote on some propositions:

- Y/N: Approve the CFO's annual report;
- Y/N: Reappoint the accounting firm;
- Y/N: Increase the CEO's salary.

- Each stockholder participates in these decisions with a **number of votes proportional to the number of stocks owned**.

Definition

A **weighted voting system** is a voting system in which

- Each voter holds a certain number of votes, his/her **weight**

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A **weighted voting system** is a voting system in which

- Each voter holds a certain number of votes, his/her **weight**
- all votes are on two-alternative (Y/N) propositions

Example

- The Kleen Car Wash Co. is owned by 4 shareholders, **A**, **B**, **C**, and **D**.

Example

- The Kleen Car Wash Co. is owned by 4 shareholders, **A**, **B**, **C**, and **D**.
- They own 40%, 30%, 20%, and 10%, respectively, of the company stock.

This is a weighted voting system where

- **A** has 4 votes
- **B** has 3 votes
- **C** has 2 votes
- **D** has 1 vote.

Question

Is it clear how many votes it should take to pass a proposition?

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Definition

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The quota depends on the situation.

- Since the total number of votes is 10, the quota should be at least 6

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- If it's less, say 5, then a proposition could be both passed and rejected.

- Since the total number of votes is 10, the quota should be at least 6
- If it's less, say 5, then a proposition could be both passed and rejected. A voting system which allows this contradiction is called **anarchy**.

It also makes no sense to have a quota bigger than 10:

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It also makes no sense to have a quota bigger than 10: nothing could ever pass! Such a system is called a **non-functional** voting system.

Since we are interested in functional systems which are not anarchic, we impose the following quota restriction:

Quota Restriction

In a weighted voting system with n voters having weights $w_1, w_2, \ldots, w_n$, the quota q is restricted by the inequalities

$$\frac{w}{2} < q \leq w$$

where $w = w_1 + w_2 + \cdots + w_n$ is the **total weight** of the system.

In our previous example:

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- The total weight is

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- The total weight is $w = 4 + 3 + 2 + 1 = 10$.

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- The total weight is $w = 4 + 3 + 2 + 1 = 10$.
- so, the quota restriction is $5 < q \leq 10$.

Careful! The bottom inequality is $<$ and the top is $\leq$.

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When the quota is

- just above half of the total weight, we have a **simple majority** voting system

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Definition

When the quota is

- just above half of the total weight, we have a **simple majority** voting system
- equal to the total weight, we have a **unanimity** voting system; here it takes all voters to say "yes" for a proposition to pass

Definition

- 2/3 of the total weight, we have a **2/3-majority** voting system

Definition

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Simple majority and 2/3-majority voting systems are common.

To simplify notation, we write a voting system in the following format:

$$[q : w_1, w_2, \dots, w_n]$$

where

- n is the number of voters

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- q is the quota.

Usually the weights are listed in decreasing order.

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- **special propositions** which deal with important things. These require a **2/3 majority** to pass.

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- **special propositions** which deal with important things. These require a **2/3 majority** to pass.
- **ordinary propositions**, which are the rest. These require only a **simple majority**.

Example

Describe the two voting systems, in the form

$$[q : w_1, w_2, \ldots, w_n]$$

Solution

Ordinary propositions *A simple majority quota is* $q = 6$*, so this system is*

$$[6 : 4, 3, 2, 1]$$

Solution

Special propositions

- $2/3$ *of* 10 *is* 6.67 *but the quota must be an integer so we take* $q = 7$.

Solution

Special propositions

- $2/3$ *of* 10 *is* 6.67 *but the quota must be an integer so we take* $q = 7$.
- *The special proposition voting system is therefore*

$$[7 : 4, 3, 2, 1]$$

Solution

Special propositions

- *$2/3$ of 10 is 6.67 but the quota must be an integer so we take $q = 7$.*
- *The special proposition voting system is therefore*

$$[7 : 4, 3, 2, 1]$$

A has enough votes to **stop** a special proposition on her own!

Definition

We say that a voter has **veto power** if he/she has enough weight to **block** a proposition on his/her own.

Note, however, that **A** does not have enough weight to **pass** a proposition all by herself.

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Definition

We say that a voter is a **dictator** if they have enough weight to pass a proposition all by themselves.

Remarks:

- A dictator must have weight $w_i \geq q > w/2$ and therefore there can be **no more than one dictator**.

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- A dictator must have weight $w_i \geq q > w/2$ and therefore there can be **no more than one dictator**.
- Voting systems with a dictator are not so interesting.
- The dictator can do whatever they want.

Example

Consider the voting system $[8 : 10, 3, 2]$.

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- It has a dictator, namely the first voter
- The other two voters have no power in the system.

For all practical purposes this system is the same as

$$[1 : 1, 0, 0].$$

Next time:

Section 1.3.1 : Coalitions and Section 1.3.2 : Critical voters; Power Index