Section 3.2.4: Iterated Two-Outcome Experiments

(In the book this is called “Two-Outcome Experiment; Repeat it!”)

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Generically, success (**S**) and failure (**F**).

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For example, Ted Williams had a batting average of .344, so the random experiment “Ted Williams at bat” with outcomes hit or no hit has probability space:

outcome	H	N
probability	0.344	0.656

Definition

When a random experiment is repeated several times (iterated), we get a Compound Random Experiment.

Example

Flip a fair coin 5 times.

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Flip a fair coin 5 times. One outcome is **HHTHT**.

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a) How many outcomes are there in the sample space?

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b) What is the probability of each outcome?

c) What is the probability of getting 2 heads and 3 tails?

Solution

a) By the product principle

- *the experiment can be broken down into 5 independent steps*
- *each step has 2 possible outcomes*

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So there are $2^5 = 32$ *outcomes in this sample space.*

Solution

b) **Remark:** *The basic random experiment (flip once) has equally likely outcomes, so the compound random experiment (flip 5 times) does too.*

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The coin is fair, so each outcome has probability $\frac{1}{32}$.

c) To find the probability of the event **E**: "2 heads and 3 tails" we use

$$\mathrm{pr}(E) = \frac{k}{n}$$

$$= \frac{\text{number of outcomes in } \mathbf{E}}{\text{number of all possible outcomes}}$$

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Need to count!

Count outcomes with exactly 2 **H**'s and 3 **T**'s.

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$$ \mathrm{pr}(E) = \frac{10}{32} = 0.3125 $$

Harder: Compound random experiments where the repeated experiment does not have equally likely outcomes.

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These can be broken down into a sequence of steps.

Example

Joe takes a 10 question multiple-choice quiz. He randomly picks one of the 5 choices in each question.

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Joe takes a 10 question multiple-choice quiz. He randomly picks one of the 5 choices in each question. Look at his sequence of successes (**S**) and failures (**F**)....

Example

a) How many different outcomes are there in the sample space?

Example

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b) What is the probability of each outcome?

Example

a) How many different outcomes are there in the sample space?
b) What is the probability of each outcome?
c) What is the probability of getting 1 correct and 9 wrong answers?

Solution

a) We write each outcome as a 10 letter sequence of S's and F's:

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To count these all, we have to pick one of 2 alternatives in each of 10 independent steps. By the product principle there are

$$\underbrace{2 \times \cdots \times 2}_{10\ times} = 2^{10} = 1,024$$

b) We may be tempted to say that the probability of each outcome is

$$\frac{1}{1,024},$$

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We need something like the product principle for probabilities in order to deal with this.

Product Principle for Probabilities

- Suppose a random experiment can be broken down into a sequence of steps, each one being a random experiment on its own.

- Assume moreover, that the probability assignments of the different steps are independent of each other.
- Then the probability of an outcome in the whole random experiment is the **product** of the probabilities of the individual steps.

By the product principle:

$$\mathrm{pr}(\textbf{SSSSFFSSFS})$$

$$\begin{aligned} &= 0.2 \times 0.2 \times 0.2 \times 0.2 \times 0.8 \\ &\quad \times 0.8 \times 0.2 \times 0.2 \times 0.8 \times 0.2 \\ &= (0.2)^7 \times (0.8)^3 \\ &= 0.0000065536 \end{aligned}$$

whereas

$$\mathrm{pr}(\textbf{FSSSFSSFSF})$$

$$\begin{aligned} &= 0.8 \times 0.2 \times 0.2 \times 0.2 \times 0.8 \\ &\quad \times 0.2 \times 0.2 \times 0.8 \times 0.2 \times 0.8 \\ &= (0.2)^6 \times (0.8)^4 \\ &= 0.0000262144. \end{aligned}$$

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(General formula coming soon)

c) Any individual outcome with 1 **S** and 9 **F**'s has probability

$$(0.2)^1 \times (0.8)^9 = 0.0268435456$$

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Selecting such an outcome amounts to selecting one place for the **S**, and placing **F**'s everywhere else.

Therefore there are

$$_{10}C_1 = 10$$

such outcomes,

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such outcomes, and the probability of the event

E: "1 correct and 9 wrong answers"

$$ \mathrm{pr}(\mathbf{E}) = 0.268435456 $$

or almost 27%.

The formula used in the last example is a special case of the general formula for the probability of a 2-outcome event repeated n times:

$$\mathrm{pr}(\mathbf{E}) = {}_nC_k \times (\mathrm{pr}(\mathbf{S}))^k \times (\mathrm{pr}(\mathbf{F}))^{n-k},$$

where **E** is the event "k successes and $n - k$ failures".

$$\mathrm{pr}(\mathbf{E}) = {}_nC_k \times (\mathrm{pr}(\mathbf{S}))^k \times (\mathrm{pr}(\mathbf{F}))^{n-k},$$

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- $\mathrm{pr}(\mathbf{S})$ is probability of a **success**

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- $\mathrm{pr}(\mathbf{S})$ is probability of a **success**
- $\mathrm{pr}(\mathbf{F})$ is probability of a **failure**

Example

A doctored coin with $\mathrm{pr}(\mathbf{H}) = 0.6$ is flipped six times. What is the probability of obtaining 2**H** and 4**T**?

Solution

We have

$$pr(\mathbf{T}) = 1 - pr(\mathbf{H}) = 0.4$$

Since $n = 6$ *and* $k = 2$, *it follows* $n - k = 4$.

Solution

Thus for the event **E***: "2***H** *and* 4**T***" we obtain:*

$$pr(\mathbf{E}) = {}_6C_2 \times (0.6)^2 \times (0.4)^4$$
$$= 15 \times 0.36 \times 0.0256$$
$$= 0.13824$$

or almost 14%.

Next time: Section 3.2.4 continued: Sum/complement principles for probabilities