Today's plan:

- Section 2.3.2: Jefferson's method (and start section 2.3.3).

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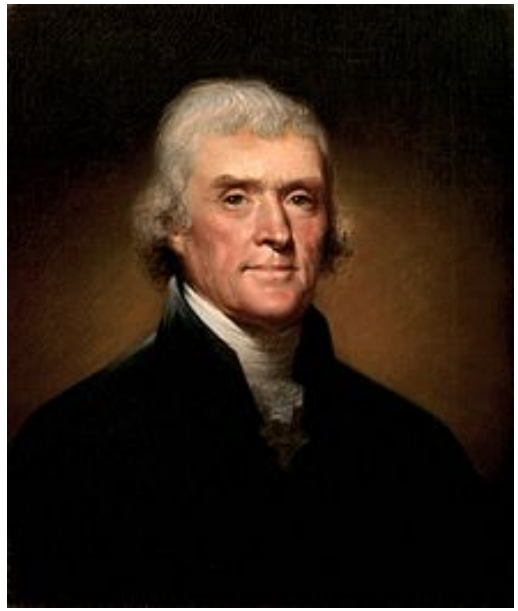
The second apportionment method, considered in 1791 by congress, was proposed by Thomas Jefferson and the Republicans.



Thomas Jefferson (1743 - 1826)

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- Instead, it uses a **"modified divisor"** *md*.

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- When md is used, there will be no surplus, just by definition.

Example

Determine the apportionment of the 100 computers to the school district with **Jefferson's method**.

Recall:

School	A	B	C	D	E	F	**Total**
Enrollment	251	379	154	228	195	217	1424

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- The goal is to get rid of the surplus, so we need to **increase** the quotas

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$$\textbf{st. quota} = \frac{\textbf{enrollment}}{\textbf{st.divisor}}$$

- To increase the quota we can **decrease** the divisor!

So let's modify the divisor from $sd = 14.24$ to $md = 13.5$ (trial-and-error).

School	A	B	C	D	E	F	Total
Enrollment	251	379	154	228	195	217	1424
No. computers:	100			Md divisor:	13.5		
Md Quota	18.593	28.074	11.407	16.889	14.444	16.074	
Md Lower	18	28	11	16	14	16	103

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- Try...

Eventually we find that $md = 13.94$ neither leaves a surplus nor produces deficit. Perfect.

School	A	B	C	D	E	F	Total
Enrollment	251	379	154	228	195	217	1424
No. computers:	100			Md divisor:	13.94		
Md Quota	18.006	27.188	11.047	16.356	13.989	15.567	
Md Lower	18	27	11	16	13	15	100

Remarks:

- Sum of the quotas is **too high** $\Rightarrow$ increase *md*

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- Sum of the quotas is **too high** $\Rightarrow$ increase *md*
- Sum of the quotas is **too low** $\Rightarrow$ decrease *md*
- There are algorithms to calculate *md*, but they are beyond the scope of this course

- We'll just use trial-and-error

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- There is more than one *md* that will work

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- There is more than one *md* that will work
- In the previous example any

$$13.929 \leq md \leq 13.944$$

works

Jefferson's Method

- Use the standard divisor to find the standard lower quotas.
- Compare the sum of lower quotas with the number of seats. If there is a surplus, reduce the divisor. If there is a deficit, increase the divisor.
- When the sum of the modified lower quotas **equals** the number of seats, we're done.
- Each state's allocation is the modified lower quota.

Back to 1791

Congress voted on the two proposals, and chose the one supported by the Federalists:

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Congress voted on the two proposals, and chose the one supported by the Federalists: a House of Representatives with $M = 120$ seats, to be apportioned using **Hamilton's method**

When the bill was sent to President Washington, he vetoed it.

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Realizing that they couldn't override the veto, congress went with the Republicans' proposal:

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Example

This apportionment was actually used for the election of 1794. Determine the number of seats allocated to each state.

- We need to apportion 105 seats

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- Our first guess for a modified divisor is just the standard divisor

$$d = \frac{3,615,920}{105} = 34,437.333$$

State	**Population**	**Exact quota**	**Lower quota**
M=105	**d**=34,437.333		
Virginia	630,560	18.310	18
Massachusetts	475,327	13.803	13
Pennsylvania	432,879	12.570	12
North Carolina	353,523	10.266	10
New York	331,589	9.629	9
Maryland	278,514	8.088	8
Connecticut	236,841	6.877	6
South Carolina	206,236	5.989	5
New Jersey	179,570	5.214	5
New Hampshire	141,822	4.118	4
Vermont	85,533	2.484	2
Georgia	70,835	2.057	2
Kentucky	68,705	1.995	1
Rhode Island	68,446	1.988	1
Delaware	55,540	1.613	1
Total	3,615,920		97

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- To increase quotas, we need to **lower the divisor**
- After some trial and error, we find that $md = 33,000$ works.

State	**Population**	**Modified Exact quota**	**Modified Lower quota**
M=105	**md**=33,000		
Virginia	630,560	19.108	19
Massachusetts	475,327	14.404	14
Pennsylvania	432,879	13.118	13
North Carolina	353,523	10.713	10
New York	331,589	10.048	10
Maryland	278,514	8.440	8
Connecticut	236,841	7.177	7
South Carolina	206,236	6.250	6
New Jersey	179,570	5.442	5
New Hampshire	141,822	4.298	4
Vermont	85,533	2.592	2
Georgia	70,835	2.147	2
Kentucky	68,705	2.082	2
Rhode Island	68,446	2.074	2
Delaware	55,540	1.683	1
Total	3,615,920		105

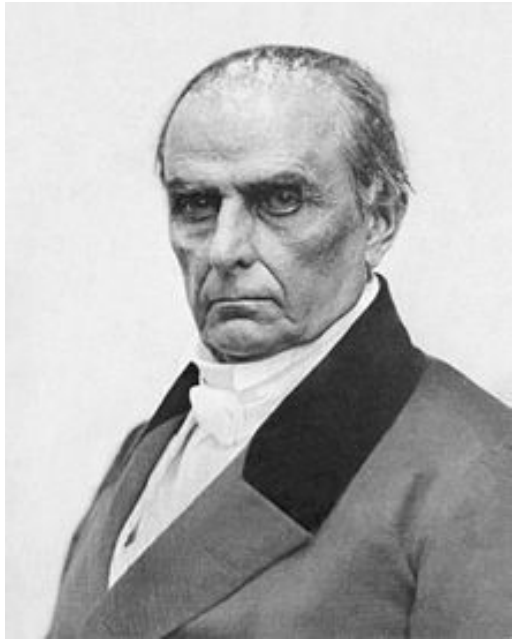
Section 2.3.3: Other Apportionment Methods.

Some History:

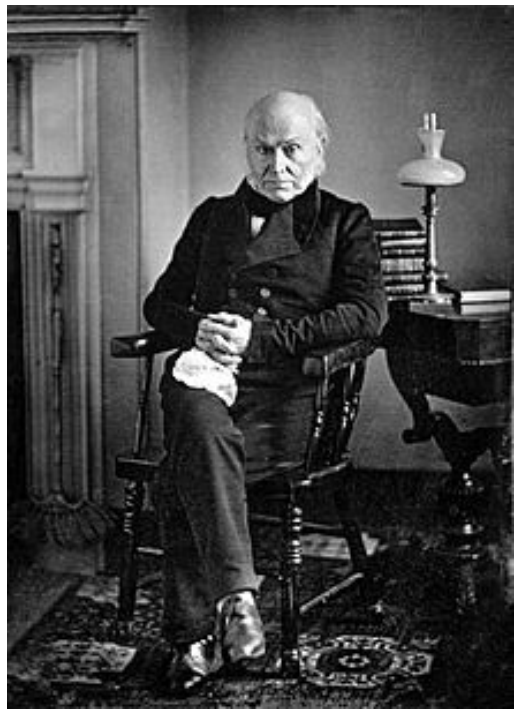
Jefferson's method was adopted in 1791, but the house kept growing and the apportionment method kept changing, throughout the 19th century and the early part of the 20th century.

Some History:

In 1832 Daniel Webster and John Quincy Adams each presented a new proposal for apportionment.



Daniel Webster (1782 - 1852)



John Quincy Adams (1767 - 1848)

More History:

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- In 1852 Hamilton's method was adopted, nearly 60 years after it was vetoed by Washington.
- In 1901 they readopted Webster's method.
- In 1941 Congress adopted a permanent method and size.

- Over time the size gradually increased from M=105 to M=433.

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- The new permanent size is M=435.

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- The new permanent size is M=435.
- The new permanent method adopted is the Huntington-Hill method.

- All three methods, **Adams' method**, **Webster's method**, and the **Huntington-Hill method** are similar to Jefferson's method, in that they use *md* to avoid a surplus

- All three methods, **Adams' method**, **Webster's method**, and the **Huntington-Hill method** are similar to Jefferson's method, in that they use *md* to avoid a surplus
- They differ in the way rounding is done

Apportionment Methods Rounding

Method	**Rounding of the Modified quota**	**Examples**
Jefferson	round down	$5.32 \to 5$ $5.98 \to 5$
Adams	round up	$5.32 \to 6$ $5.98 \to 6$
Webster	round to nearest integer	$5.32 \to 5$ $5.98 \to 6$ $5.5 \to 6$
Huntington-Hill	according to geometric mean $\sqrt{n(n+1)}$	$5.32 \to 5$ $5.485 \to 6$ $\sqrt{5 \cdot 6} = 5.477$

Exact Quota	Jefferson Lower	Adams Upper	Webster Nearest	Hunt.-Hill Geom. mean
4.000	4	4	4	4
4.178	4	5	4	4
4.475	4	5	4	5
4.500	4	5	5	5
4.615	4	5	5	5

$$\sqrt{4 \times 5} \approx 4.472$$

- All methods go by trial and error to find an *md* that leaves no surplus nor deficit

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- If there's a surplus the divisor is reduced
- if there's is a deficit the divisor is increased

Example

- The Archipelagic Confederation consists of four islands, with a government cabinet consisting of 18 members

Example

- The Archipelagic Confederation consists of four islands, with a government cabinet consisting of 18 members
- The seats in the cabinet are assigned to the four islands based on their population.

Example (Continued)

The population of islands is:

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**
Population	1,205,000	163,000	267,000	165,000

Example (Continued)

The population of islands is:

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**
Population	1,205,000	163,000	267,000	165,000

Determine the apportionment of the cabinet seats to the four islands using Adams' method, Webster's method, and the Huntington-Hill method.

- As with Jefferson's method, we start with the standard divisor, and go from there to find the modified divisor that works.

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- The standard divisor is

$$d = \frac{1,800,000}{18} = 100,000$$

Adams' method

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M $= 18$				**d** $= 100{,}000$	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	13	2	3	2	20 Too high!

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M = 18				**d** = 100,000	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	13	2	3	2	20 Too high!
M = 18				**md** = 105,000	
Md Exact q.	11.48	1.55	2.54	1.57	
Rounded q.	12	2	3	2	19 Too high!

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M = 18				**d** = 100,000	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	13	2	3	2	20 Too high!
M = 18				**md** = 105,000	
Md Exact q.	11.48	1.55	2.54	1.57	
Rounded q.	12	2	3	2	19 Too high!
M = 18				**md** = 110,000	
Md Exact q.	10.95	1.48	2.43	1.5	
Rounded q.	11	2	3	2	18 Just right!

Webster's method

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M $= 18$				**d** $= 100{,}000$	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	12	2	3	2	19 Too high!

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M = 18				**d** = 100,000	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	12	2	3	2	19 Too high!
M = 18				**md** = 110,000	
Md Exact q.	10.955	1.482	2.427	1.500	
Rounded q.	11	1	2	2	16 Too low!

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M = 18				**d** = 100,000	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	12	2	3	2	19 Too high!
M = 18				**md** = 110,000	
Md Exact q.	10.955	1.482	2.427	1.500	
Rounded q.	11	1	2	2	16 Too low!
M = 18				**md** = 105,000	
Md Exact q.	11.476	1.552	2.543	1.571	
Rounded q.	11	2	3	2	18 Just right!

Huntington-Hill's method

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
M $= 18$				**d** $= 100{,}000$	
Exact q.	12.050	1.630	2.670	1.650	
Rounded q.	12	2	3	2	19 Too high!

Island	**Arisa**	**Beruga**	**Crispa**	**Daria**	**Total**
Pop.	1,205,000	163,000	267,000	165,000	1,800,000
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Next time: Section 2.3.4.: Problems with Apportionment