Math 605: Final Exam 2017 Spring

Send an electronic version to zshang@binghamton.edu before 5pm May 15, 2017. Late submission will NOT be counted.

(1) (5 points) Suppose that $Y \sim N(\theta, I_n)$. Show that $\widehat{\theta} = Y$ is a minimax estimator of θ.

(2) (5 points) Show that the positive-part James-Stein estimator has smaller estimation risk than the James-Stein estimator. Find the ratio of these risks as the number of observations diverges.

(3) (a) (3 points) Suppose $X|\theta \sim N(\theta, I_p)$, where $\theta \in \mathbb{R}^p$ is unknown mean vector. The aim is to estimate θ. Consider the following general form of James-Stein estimator

$$\widehat{\theta}_a = \left(1 - \frac{a}{\|X\|_2^2}\right) X, \quad 0 < a < 2(p-2).$$

Show that $E_\theta \|\widehat{\theta}_a - \theta\|_2^2 < E_\theta \|X - \theta\|_2^2$ for any θ.

(b) Consider a linear regression $Y_i = X_i\beta + \epsilon_i$, $i = 1, \ldots, n$, where $\beta \in \mathbb{R}^p$ with p fixed and $\epsilon_i \stackrel{iid}{\sim} N(0, 1)$.

(b1) (2 points) Derive a James-Stein estimator $\widehat{\beta}^{JS}$ for β.

(b2) (4 points) Show that $\widehat{\beta}^{JS}$ dominates the ordinary least square estimator in terms of L^2 risk under any $\beta \in \mathbb{R}^p$.

(b3) (4 points) Can you further improve $\widehat{\beta}^{JS}$?

(4) Consider again the Bayesian variable selection problem with the second type of spike & slab prior, i.e., the following hierarchical model

$$Y_i|X_i, \beta_1, \ldots, \beta_p, \sigma^2 \stackrel{ind}{\sim} N(\beta_1 X_{i1} + \cdots + \beta_p X_{ip}, \sigma^2), \quad i = 1, \ldots, n,$$

with prior distributions

$$\begin{aligned} \beta_j|\gamma_j, \sigma^2 &\stackrel{ind}{\sim} (1-\gamma_j)\delta_0 + \gamma_j N(0, c\sigma^2) \\ \gamma_1, \ldots, \gamma_p &\stackrel{ind}{\sim} \text{Bernoulli}(p) \\ \sigma^2, \tau_1^2, \ldots, \tau_p^2 &\stackrel{iid}{\sim} IG(a, b), \end{aligned}$$

where $a, b, c > 0, 0 < p < 1$ are hyperparameters.

(a) (4 points) Derive (in full details) an empirical Bayes approach for estimating $\beta_1, \ldots, \beta_p$.

(b) (3 points) Examine your EB approach via the pseudo data provided in my website. Show full details.

(5) (10 points) Approximate the following integral $I(\gamma)$ as $\gamma \to \infty$:

$$I(\gamma) = \int_{x_1,\ldots,x_n \in \mathbb{S}^1} \exp\left(\gamma \sum_{i,j=1}^{n} a_{ij} x_i^T x_j\right) dx_1 \cdots dx_n,$$

where S^1 is the unit circle and $A = [a_{ij}]_{i,j=1}^n$ is any $n \times n$ symmetric real matrix.