Homework 10

Do the problems on **Webwork** and upload the following problems to Gradescope before 8 am on April 3rd.
When you upload your assignment, mark the page on which your solution to each problem starts, or upload each problem individually.

Homework should be written neatly and clearly explained. Include your name and id number in the top right corner of your homework.

Problem 1. Which of the following pairs of random variables are independent?

(a) Let X_1 and X_2 have joint pdf

$$f_{X_1,X_2}(x_1,x_2) = \frac{3}{4}x_1^2(1-x_2) \text{ for } 0 \le x_1 \le 2, 0 \le x_2 \le 1$$

and 0 otherwise.

(b) Let Y_1 and Y_2 have joint pdf

$$f_{Y_1,Y_2}(y_1,y_2) = 3y_1y_2$$

on the region bounded by the lines $y_2 = 0$, $y_1 = y_2$ and $y_1 + y_2 = 2$.

(c) Let Z_1 and Z_2 have joint pdf

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{6\pi}e^{-\left(\frac{z_1^2}{4}+\frac{z_2^2}{9}\right)}$$

for all z_1, z_2.

Hint: You don't actually need to compute the marginal distributions. Though it is good practice.

There is a theorem about showing random variables are independent if they are supported on a (possibly infinite) rectangle and there pdf factorizes into a product of two functions.

To show rv are not independent, you just need to show there is some point where $f_{X,Y}(x,y) \ne f_X f_Y(y)$, and all the function are continuous in a neighborhood of that point. A good thing to try is, to find a point where $f_{X,Y}(x,y)$ is zero and f_X, $f_Y(y)$ are not. This technique doesn't always work, but it's a good place to start, if you can't find a point where $f_{X,Y}(x,y)$ does not equal $f_X f_Y(y)$ because one is zero, then you do actually need to compute the marginal pdf.

Problem 2. A person has a highly contagious disease. The number of people they meet each day is a Poisson random variable with mean 5. They infect each person they meet with with probability 1/3, independently.

Let Y be the number of people they meet and X be the number they infect.

(a) Conditioned on the event that $\{Y = n\}$, what is the pmf of X? (Your answer should involve n).

(b) What is the joint distribution of X and Y? (Don't forget the bounds.)

(c) Show that X is a Poisson random variable. What is $\mathbb{E}[X]$?

Hint: part c
After expanding your joint pmf you should something like:

$$\mathbb{P}(\{X = k\}, (\{Y = n\},) = \frac{1}{(n-k)!k!} p^k (1-p)^{n-k} e^{-\lambda} \lambda^n$$

for $0 \le k \le n = 0, 1, 2 \ldots$. (DON'T FORGET THE DOMAIN. Note figuring out how to get here is an important part of the problem and requires work and some simplifications, your p and λ will be actual numbers)

To compute the marginal you want to do the sum:

$$\mathbb{P}(\{X = k\}) = \sum_{n=k}^{\infty} \frac{1}{(n-k)!k!} p^k (1-p)^{n-k} e^{-\lambda} \lambda^n$$

(NOTE: the lower bound starts at $n = k$, because that is where the formula for pmf holds) To compute a sum like this first pull out all the terms that don't depend on n

$$\frac{1}{k!} p^k \lambda^k e^{-\lambda} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} (1-p)^{n-k} \lambda^{n-k}$$

This sum would look much nicer if the sum started at 0, so make the change of index $m = n - k$, and sum over m instead. So the sum will start at $m = 0$ and everywhere you see n replace it with $m + k$. (Remember we're trying to compute the pmf of X at k. So we're treating k as a fixed number).

Then hopefully you have a nicer looking infinite sum that you can compute. In general, when you are using Poisson random variables, the Taylor Series for e^x might be useful.