On the Free Convolution with a Semi-circular Distribution

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Abstract. We give a formula for the density of the free convolution of an arbitrary probability measure with a semi-circular distribution. We use this formula to establish a certain number of regularity properties of the measures obtained in this way.

Introduction. In recent years, a new type of harmonic analysis of measures on the real line has been developped, centered around the definition of free convolution of measures, introduced by D. Voiculescu. Many classical results in the theory of addition of independent random variables have their counterpart in this new theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchine formula, etcetera. We refer to [VDN] for an introduction to these topics. One of the main problems in dealing with free convolution is that its definition is rather indirect, and one does not get easily formulas for computing densities of free convolution of measures. In this note, we propose such a formula when one of the measures is a semi-circular distribution, and we use it to prove certain regularizing properties of the operation of taking the free convolution with a semi-circular distribution. The semi-circular distribution is the analogue in free probability theory of the gaussian distribution, so that what we are investigating is an explicit formula for the solution to the free analogue of the heat equation. In particular we prove that the measures obtained by free convolution with a semi-circular distribution of variance t have always a continuous density whose cube is Lipschitz, with Lipschitz constant less than $3(4\pi^3t^2)^{-1}$, and we also give optimal bounds on the density of such measures, when the starting measure has a bounded density. We also give a formula for the Hilbert ransform of the density.

This topic has several connections with other areas of mathematics. The free convolution has an interpretation in terms of spectra of large random matrices, see [V2], [S]. The convolution with a semi-circular distribution plays an important role in the theory of free entropy developped by D. Voiculescu see [V3]

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and the subsequent papers, which has had some remarkable applications to the theory of operator algebras. Let us also mention a connection with nonlinear partial differential equations, if μ_t is the convolution of the measure μ with a semi-circular distribution of variance t, then the Cauchy transforms satisfy the following complex Burger's equation

$$\frac{\partial G_{\mu_t}(z)}{\partial t} + G_{\mu_t}(z)\frac{\partial G_{\mu_t}(z)}{\partial z} = 0 .$$

This is the free analogue of the heat equation, see e.g. [VDN].

We intend to use these results in further work on the analogues in free probability of the diffusion processes on the real line, which have been shown to occur in some aspects of matrix models arising in quantum field theory (see [D]).

We shall recall the definition of free convolution in Section 1. In Section 2 we will make some remarks on the connections between this definition and the theory of conformal mappings. We shall use these remarks in Section 3 to give the explicit formula for the density of the convolution of an arbitrary measure with a semi-circular distribution. Finally we shall apply this formula in Section 4 to give the regularity results announced above.

1. Definition of free convolution. Let μ be a probability measure on $\mathbb{R}$. The Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z - x}$$

is an analytic function on the complex upper half plane $\mathbb{C}^+$. There exists a domain

$$D_{\alpha,\beta} = \{u + iv \in \mathbb{C} \mid |u| < \alpha v \ \ v > \beta\}$$

for some α, $\beta > 0$ on which G_μ is univalent. The image $G_\mu(D_{\alpha,\beta})$ contains a domain

$$\Gamma_{\gamma,\delta} = \{u + iv \mid |u| < -\gamma v;\ 0 > v > -\delta\}$$

for some γ, $\delta > 0$. Let K_μ be the inverse function of G_μ defined on $\Gamma_{\gamma,\delta}$, and

$$R_\mu(z) = K_\mu(z) - \frac{1}{z} .$$

Then given two probability measures λ and μ, there exists a unique probability measure ν such that

$$R_\nu = R_\lambda + R_\mu$$

on a domain $\Gamma_{\gamma,\delta}$ where these three functions are defined. The measure ν is denoted $\lambda \boxplus \mu$ and is called the free convolution of λ and μ. We refer to [VDN] for the operator theoretic background of this definition. The free convolution was first defined by D. Voiculescu for measures with compact support [V1], extended to measures with finite variance by H. Maassen [M], and then to arbitrary measures in [BV].

2. Free convolution and conformal mapping. Our analysis of the free convolution with a semi-circular distribution will be based on the following observations. We shall denote by μ and λ two probability measures on $\mathbb{R}$ and by ν their free convolution.

Lemma 1. *There exists a domain $D_{\alpha,\beta}$ on which one has*

$$G_\mu(z) = G_\nu\big(z + R_\lambda(G_\mu(z))\big).$$

Proof. By the definition of ν one has

$$K_\nu = K_\mu + R_\lambda$$

on some domain $\Gamma_{\gamma,\delta}$. We can choose a domain of the form $D_{\alpha,\beta}$ which is mapped into $\Gamma_{\gamma,\delta}$ by G_μ (see [BV], Section 5) and thus

$$K_\nu(G_\mu(z)) = z + R_\lambda(G_\mu(z))$$

for $z \in D_{\alpha,\beta}$. Taking the image by G_ν of both sides, we get the result. □

It is proved in [B] Theorem 3.1, that there exists an analytic map $F : \mathbb{C}^+ \to \mathbb{C}^+$ such that $F(z)/z \to 1$ as $z \to \infty$ with $z \in D_{\alpha,\beta}$, for every such domain, and such that

$$G_\nu(z) = G_\mu(F(z))$$

for all $z \in \mathbb{C}^+$. We shall assume that the function $H(z) = z + R_\lambda(G_\mu(z))$, defined in some domain $D_{\alpha,\beta}$, can be extended analytically to $\mathbb{C}^+$. This hypothesis is fulfilled, in particular, if the measure λ is freely infinitely divisible, in which case R_λ has an analytic extension to the lower half plane $\mathbb{C}^-$ (see [BV]). Let $\Omega_{\mu,\lambda}$ be the connected component of the set $H^{-1}(\mathbb{C}^+)$ which contains iy for large y. Under the above hypotheses, we have the following extension of (i) and (ii) of Proposition 5.12 in [BV] (which corresponds to the case $\mu = \delta_0$).

Proposition 1. *The function F is a one to one conformal mapping, with image $\Omega_{\mu,\lambda}$. Its inverse is the restriction of H to $\Omega_{\mu,\lambda}$.*

Proof. In some domain $D_{\alpha,\beta}$, one has, thanks to Lemma 1,

$$G_\nu(z) = G_\mu(F(z)) = G_\nu(H(F(z)))\,.$$

If the domain $D_{\alpha,\beta}$ is chosen small enough, then G_ν is univalent on it and $H \circ F$ maps this domain into a domain of univalence of G_ν, hence one has $H(F(z)) = z$ in this domain. This extends by analytic continuation to $z \in \mathbb{C}^+$, hence F is univalent and $F(\mathbb{C}^+) \subset \Omega_{\mu,\nu}$. On the other hand $F(H(F(z))) = F(z)$, hence $F(H(z)) = z$ for $z \in F(\mathbb{C}^+)$ and by analytic continuation, this holds for $z \in \Omega_{\mu,\lambda}$, and the conclusion follows easily. □

Remark. It is not true in general that the map F is univalent, take e.g. $\mu = \delta_0$, then $\nu = \lambda$ and $F = 1/G_\lambda$ which is not univalent in general.

3. Free convolution with a semi-circular distribution. In the case where λ is a semi-circular distribution, one can give an explicit description of the set $\Omega_{\mu,\lambda}$, as we shall see below.

The semi-circular distribution of variance t is the probability measure with density

$$\lambda_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,,dx$$

on the interval $[-2\sqrt{t}, 2\sqrt{t}]$. One can compute

$$G_{\lambda_t}(z) = \frac{z - \sqrt{z^2 - 4t}}{2t}$$

on the upper half plane $\mathbb{C}^+$, where the branch of the square root on $\mathbb{C} \setminus \mathbb{R}_+$ is such that $\sqrt{-1} = i$. One has then

$$R_{\lambda_t}(z) = tz\,,$$

which is an analytic function in $\mathbb{C}^-$. We now fix the variance t of the semi-circular distribution. Let

$$U_t = \left\{u \in \mathbb{R} \,|\, \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2} > \frac{1}{t}\right\},$$

then U_t is an open set. Define the function $v_t : \mathbb{R} \to \mathbb{R}_+$ as

$$v_t(u) = \inf\left\{v \geq 0 \,|\, \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v^2} \leq \frac{1}{t}\right\}.$$

Since

$$v \mapsto \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2+v^2}$$

is a decreasing function, the following statements are easy to check.

Lemma 2. *The function v_t is continuous on $\mathbb{R}$, analytic on the open set U_t, and one has $U_t = \{u \in \mathbb{R} \mid v_t(u) > 0\}$. Furthermore for all $u \in \mathbb{R}$*

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2+v_t(u)^2} \leq 1/t$$

with equality if $v_t(u) > 0$.

Let

$$\Omega_{\mu,t} = \{u+iv \in \mathbb{C}^+ | v > v_t(u)\}$$

be the part of the upper half plane which is above the graph of v_t. Then $\Omega_{\mu,t}$ is a simply connected region whose boundary is the graph of v_t, and one has

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|z-x|^2} < \frac{1}{t}$$

for all $z \in \Omega_{\mu,t}$.

Lemma 3. *The function G_μ has a continuous extension to $\overline{\Omega_{\mu,t}}$, which is Lipschitz, with Lipschitz constant $\leq 1/t$, and one has $|G_\mu(z)|^2 \leq 1/t$ for $z \in \overline{\Omega_{\mu,t}}$.*

Proof. Indeed, one has

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|z-x|^2} \leq \frac{1}{t}$$

for $z \in \Omega_{\mu,t}$, hence, by Fatou's Lemma, this inequality still holds for $z \in \partial\Omega_{\mu,t}$, hence the integral $\int_{\mathbb{R}} d\mu(x)/(z-x)$ converges for all $z \in \partial\Omega_{\mu,t}$, and one has, for all z_1, $z_2 \in \overline{\Omega_{\mu,t}}$

$$\left|\frac{G_\mu(z_1)-G_\mu(z_2)}{z_1-z_2}\right| = \left|\int_{\mathbb{R}} \frac{d\mu(x)}{(z_1-x)(z_2-x)}\right|$$
$$\leq \sqrt{\int_{\mathbb{R}} \frac{d\mu(x)}{|z_1-x|^2} \int_{\mathbb{R}} \frac{d\mu(x)}{|z_2-x|^2}} \leq \frac{1}{t}.$$

By Cauchy-Schwarz inequality again, one has

$$|G_\mu(z)|^2 \leq \int_{\mathbb{R}} \frac{d\mu(x)}{|z-x|^2} \leq \frac{1}{t}. \qquad \square$$

We shall still denote by G_μ the continuous extension obtained in Lemma 3.

Lemma 4. *The map $H_t(z) = z + tG_\mu(z)$ is a homeomorphism from $\overline{\Omega_{\mu,t}}$ to $\mathbb{C}^+ \cup \mathbb{R}$ which is conformal from $\Omega_{\mu,t}$ onto $\mathbb{C}^+$.*

Proof. The map H_t is analytic on $\Omega_{\mu,t}$ and continuous on $\overline{\Omega_{\mu,t}}$ by Lemma 3. Let us prove that it is univalent on $\overline{\Omega_{\mu,t}}$. Indeed assume that $z_1 + tG_\mu(z_1) = z_2 + tG_\mu(z_2)$ for some $z_1 \neq z_2 \in \overline{\Omega_{\mu,t}}$, then one has

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(z_1 - x)(z_2 - x)} = \frac{1}{t}.$$

One has thus equality in the Cauchy-Schwarz inequality in the proof of Lemma 3, and this implies that for some $\lambda \in \mathbb{C}$ one has $|z_1 - x|^2 = \lambda(z_1 - x)(z_2 - x)$ for μ-almost all x. If μ is not reduced to a Dirac measure, since z_1 and z_2 have nonnegative imaginary parts, this implies that $z_1 = z_2$. If μ is a Dirac measure, the claim can be checked directly. For all $z \in \overline{\Omega_{\mu,t}}$ one has

$$\Im(H_t(z)) = \frac{1}{2i}(z - \overline{z}) \left(1 - t \int_{\mathbb{R}} \frac{d\mu(x)}{|z-x|^2}\right).$$

It follows that $H_t(\Omega_{\mu,t}) \subset \mathbb{C}^+$ since $\int_{\mathbb{R}} d\mu(x)/|z-x|^2 < 1/t$ on $\Omega_{\mu,t}$. On the other hand, by Lemma 2, for $z \in \partial\Omega_{\mu,t}$ one has either $z = \overline{z}$ or

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|z-x|^2} = \frac{1}{t},$$

hence $H_t(\partial\Omega_{\mu,t}) \subset \mathbb{R}$. The image of $\overline{\Omega_{\mu,t}}$ is a simply connected region included in $\mathbb{C}^+ \cup \mathbb{R}$ whose boundary is included in $\mathbb{R}$, hence its image must be the whole of $\mathbb{C}^+ \cup \mathbb{R}$. $\square$

Let us denote by $F_t : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{\mu,t}}$ the inverse function of H_t. Let μ_t be the measure $\mu \boxplus \lambda_t$. Using Proposition 1, we see that, with the notations of Section 1, $\Omega_{\mu,t}$ is equal to Ω_{μ,λ_t} and we obtain the following result:

Proposition 2. *For all $z \in \mathbb{C}^+$ one has $G_{\mu_t}(z) = G_\mu(F_t(z))$.*

The map F_t is continuous on $\mathbb{C}^+ \cup \mathbb{R}$, with values in $\overline{\Omega_{\mu,t}}$ so by Lemma 3

Corollary 1. *The Cauchy transform G_{μ_t} has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$.*

Using the inversion formula for Cauchy transforms, we get

Corollary 2. *The measure μ_t has a density given by the formula*

$$p_t(x) = \frac{1}{\pi}\Im(G_\mu(F_t(x)).$$

The Hilbert transform of this density is

$$Hp_t(x) = \frac{1}{\pi}\Re(G_\mu(F_t(x)).$$

Let us give a more explicit way of computing the density. Note that the map $u \mapsto u + iv_t(u)$ is a homeomorphism from $\mathbb{R}$ onto $\partial\Omega_{\mu,t}$.

Corollary 3. *Let*

$$\psi_t(u) = H_t(u + iv_t(u)) = \Re(H_t(u + iv_t(u))) = u + t\int_{\mathbb{R}} \frac{(u-x)d\mu(x)}{(u-x)^2 + v_t(u)^2},$$

then $\psi_t : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and at the point $\psi_t(u)$ the measure μ_t has a density given by

$$p_t(\psi_t(u)) = \frac{v_t(u)}{\pi t}.$$

Furthermore, the Hilbert transform of the density is given, at the point $\psi_t(u)$, by the formula

$$Hp_t(\psi_t(u) = \frac{1}{t}(u - \psi_t(u)) = \int_{\mathbb{R}} \frac{(u-x)d\mu(x)}{(u-x)^2 + v_t(u)^2}.$$

4. Regularity properties of the free convolution. We shall infer from the preceding formulas some regularizing properties of the free convolution with a semi-circular distribution. First, we shall study the support of the measure μ_t.

Proposition 3. *The support of the measure μ_t is the closure of its interior, furthermore the number of connected components of this interior is a non-increasing function of t.*

Proof. By Corollary 3, the measure μ_t has a continuous density, which yields the first assertion. One also sees that the interior of the support of μ_t is the image of U_t by the homeomorphism ψ_t. The open sets U_t form an increasing family. In order to prove the second assertion, it is enough to see that the number of connected components of U_t is non-increasing, and this is a consequence of the fact that

$$u \mapsto \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2}$$

cannot have a local maximum in any open interval $]\alpha,\beta[\, \subset U_t^c$. Indeed, let $]\alpha,\beta[$ be such an interval, then for any $\big]a,b\big[\subset \big]\alpha,\beta\big[$, one has

$$\frac{1}{t} \geq \int_a^b \frac{d\mu(x)}{(u-x)^2} \geq \frac{\mu\big(\big]a,b\big[\big)}{(b-a)^2}.$$

This estimate easily implies that μ does not charge the interval $]\alpha,\beta[$, thus the function $u \mapsto \int_{\mathbb{R}} d\mu(x)/(u-x)^2$ is strictly convex in this interval, and the result follows. □

The support of the measure μ_t is the closure of the image of U_t by the map ψ_t, and U_t is an increasing family of open sets, but it is not true that the support of μ_t is increasing with t. Indeed, consider for example the measure

$$\mu = \alpha\delta_{-1} + (1-\alpha)4x^3 dx 1_{[0,1]}(x),$$

then it is easy to see, using the formula for ψ_t, that for t small enough the support of the measure μ_t is

$$[-1-a(t),\ -1+b(t)] \cup \left[\frac{7\alpha-4}{3}t,\ 1+c(t)\right],$$

where a, b, c are positive and continuous with $a(0)=b(0)=c(0)=0$. If $\alpha > \frac{4}{7}$, we see that the support of μ_t does not increase with t for t small.

We now turn to the analyticity of the density.

Lemma 5. *For any $u \in U_t$ one has*

$$\psi_t'(u) \geq \frac{2}{t} v_t(u)^2 (1 + v_t'(u)^2).$$

Proof. One has

$$\psi_t(u) = u + t\int_{\mathbb{R}} \frac{(u-x)d\mu(x)}{(u-x)^2 + v_t(u)^2}$$

so that for all $u \in U_t$

$$\psi_t'(u) = 2 - 2t \int_{\mathbb{R}} \frac{(u-x)^2 + v_t(u)v_t'(u)(u-x)}{((u-x)^2 + v_t(u)^2)^2} d\mu(x)$$
$$= 2t \int_{\mathbb{R}} \frac{v_t(u)^2}{((u-x)^2 + v_t(u)^2)^2} d\mu(x) - 2t \int_{\mathbb{R}} \frac{v_t(u)v_t'(u)(u-x)}{((u-x)^2 + v_t(u)^2)^2} d\mu(x)$$

since

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_t(u)^2} = \frac{1}{t}$$

for all $u \in U_t$. On the other hand, differentiating this last equality yields

$$-2 \int_{\mathbb{R}} \frac{(u-x)}{((u-x)^2 + v_t(u)^2)^2} d\mu(x) = 2v_t(u)v_t'(u) \int_{\mathbb{R}} \frac{d\mu(x)}{((u-x)^2 + v_t(u)^2)^2} .$$

Replacing in the equation above, we get

$$\psi_t'(u) = 2tv_t(u)^2 \int_{\mathbb{R}} \frac{d\mu(x)}{((u-x)^2 + v_t(u)^2)^2} + 2tv_t(u)^2 v_t'(u)^2 \int_{\mathbb{R}} \frac{d\mu(x)}{((u-x)^2 + v_t(u)^2)^2} .$$

Applying Hölder's inequality, we obtain

$$\int_{\mathbb{R}} \frac{d\mu(x)}{((u-x)^2 + v_t(u)^2)^2} \geq \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_t(u)^2} \right)^2 = \frac{1}{t^2}$$

and we get

$$\psi_t'(u) \geq \frac{2}{t} v_t(u)^2 (1 + v_t'(u)^2) . \qquad \square$$

From this estimate we deduce that ψ_t' does not vanish on U_t, and thus its inverse $\psi_t^{(-1)}$ is analytic on the support of μ_t.

Corollary 4. *The density p_t is analytic on the set $\{x \in \mathbb{R} \mid p_t(x) > 0\}$.*

Proof. This follows from Corollary 3 and the fact that ψ_t' does not vanish on U_t, by Lemma 5. $\square$

We shall apply this estimate to the regularity of the density p_t near its zeros.

Proposition 4. *For every x such that $p_t(x) > 0$, one has*

$$|p_t^2(x) p_t'(x)| \le \frac{1}{4\pi^3 t^2}\,.$$

Proof. At the point $x = \psi_t(u)$ for $u \in U_t$, the value of the quantity to be estimated is

$$\frac{|v_t(u)^2 v_t'(u)|}{\pi^3 t^3 \psi_t'(u)} \le \frac{|t v_t'(u)|}{2\pi^3 t^3 (1 + v_t'(u)^2)} \le \frac{1}{4\pi^3 t^2}\,. \qquad \square$$

Looking at the case of $\mu = \delta_0$ and $\mu_t = \lambda_t$, we see that the bound given in Proposition 4 is attained.

Corollary 5. *For all $x \in \mathbb{R}$ one has*

$$p_t(x) \le \left(\frac{3}{4\pi^3 t^2} |x - x_0| \right)^{1/3},$$

where x_0 is the closest point to x, in the complement of the interior of the support of μ_t.

The occurrence of the exponent $\frac{1}{3}$ in the preceding Corollary is surprising at first sight, since it is worse than the $\frac{1}{2}$ occuring at the edge of the spectrum of the semi-circular distribution. We shall see that in fact the exponent $\frac{1}{3}$ gives the right singularity of the density at a point where two components of the set U_t merge into one. Let us consider a finite interval $]a, b[$ which is a connected component of the complement of the support of μ. The function $u \mapsto \int_{\mathbb{R}} d\mu(x)/(u - x)^2$ is strictly convex and finite on $]a, b[$, hence it is either increasing, decreasing, or it reaches its minimum at some unique point $u_0 \in \,]a, b[$. Let us assume we are in this last case, then $\int_{\mathbb{R}} d\mu(x)/(u_0 - x)^3 = 0$ and the open set U_t has two successive connected components $]f(t), a(t)[$ and $]b(t), g(t)[$, for

$$t \le t_0 = \frac{1}{\displaystyle\int_{\mathbb{R}} \frac{d\mu(x)}{u_0 - x)^2}}$$

such that $a(t)$ and $b(t)$ are continuous functions, $a(t) > a$ is increasing, $b(t) < b$ is decreasing. At the time t_0, one has $a(t_0) = b(t_0) = u_0$, and after t_0, the two components merge into one.

At $t = t_0$, one has $v_t(u_0) = 0$ and $v_t(u) > 0$ on $]a, b[\setminus \{u_0\}$. Since

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_t(u)^2} = \frac{1}{t}$$

for $u \in]a, b[$, in a neighbourhood of u_0 one has

$$\frac{1}{t} = \int_{\mathbb{R}} \frac{d\mu(x)}{(u_0 - x)^2} + (3(u-u_0)^2(1+o(1)) - v_t(u)^2(1+o(1))) \int_{\mathbb{R}} \frac{d\mu(x)}{(u_0-x)^4}\,.$$

This implies that

$$v_t(u)^2 = 3(u-u_0)^2(1+o(1))$$

and similarly one has

$$v_t'(u)^2 = 3 + o(1)\,.$$

From the computation in Lemma 5, we have

$$\psi_t'(u) = 2tv_t(u)^2(1+v_t'(u)^2) \int_{\mathbb{R}} \frac{d\mu(x)}{((u-x)^2 + v_t(u)^2)^2}\,,$$

so that in the same neighbourhood of u_0,

$$\psi_t'(u) = 24t(u-u_0)^2 \int_{\mathbb{R}} \frac{d\mu(x)}{(u_0-x)^4}(1+o(1))\,.$$

From this we get

$$\psi_t(u) = \psi_t(u_0) + 8t \int_{\mathbb{R}} \frac{d\mu(x)}{(u_0-x)^4}(u-u_0)^3(1+o(1))\,,$$

and inverting this, we obtain

$$p_t(x) = \frac{v_t(\psi_t^{(-1)}(x))}{\pi t} = \frac{1}{\pi t}\left(8t \int_{\mathbb{R}} \frac{d\mu(x)}{(u_0-x)^4}(u-u_0)^3\right)^{-1/3} |x-x_0|^{1/3}(1+o(1))$$

in a neighbourhood of $x_0 = \psi_t(u_0)$.

A similar analysis would yield

$$p_t(x) = \lambda |a(t) - x|^{1/2}(1+o(1))$$

for $x \in]a(t)-\varepsilon,\ a(t)[$, with $t < t_0$, and a corresponding result at $b(t)$.

We shall now investigate L^∞ bounds on the density. Using Proposition 2 and Lemma 3, we obtain $|G_{\mu_t}(z)|^2 \le 1/t$ for $z \in \mathbb{C} \cup \mathbb{R}$ (actually this bound could also be obtained directly from the fact G_{μ_t} is subordinated to G_{λ_t}), so that $p_t(x)^2 + Hp_t(x)^2 \le 1/t$ for $x \in \mathbb{R}$. We can get a more accurate upper bound for the density alone however, as we will see.

Definition. For K, $t > 0$, define $\Delta(K,t)$ as the unique solution v of

$$\frac{2K}{v}\arctan\left(\frac{1}{2Kv}\right) = \frac{1}{t}\,.$$

Lemma 6. *For all $K,t > 0$, one has*

$$\Delta(t,K) \le \frac{\pi t K}{1 + \dfrac{2}{\pi}\arctan(2\pi K^2 t)}\,.$$

Proof. One has $\Delta(t,K) = \Gamma\pi t K$ where $\Gamma \in [0,1]$ is the unique solution of

$$\rho = \frac{2}{\pi}\arctan\left(\frac{1}{2K^2\pi t\rho}\right).$$

Since the function $\rho \mapsto (2/\pi)\arctan\big(1/(2K^2\pi t\rho)\big)$ is convex, we have

$$\frac{2}{\pi}\arctan\left(\frac{1}{2K^2\pi t\rho}\right) \le 1 - \rho + \rho\frac{2}{\pi}\arctan\left(\frac{1}{2K^2\pi t}\right).$$

for $\rho \in [0,1]$ and so Γ is less than the solution of

$$\rho = 1 - \rho + \rho\frac{2}{\pi}\arctan\left(\frac{1}{2K^2\pi t}\right). \qquad \square$$

Proposition 5. *Suppose that μ has a density bounded above by K, then the density of μ_t is bounded above by $(1/\pi t)\Delta(t,K)$.*

Proof. By a simple rearrangement argument, for all $u \in \mathbb{R}$ one has

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v^2} \le K\int_{u-1/2K}^{u+1/2K} \frac{dx}{(u-x)^2 + v^2} = \frac{2K}{v}\arctan\left(\frac{1}{2Kv}\right).$$

Since both sides are decreasing functions of v, it follows that $v_t(u)$ is less than the solution of

$$\frac{2K}{v}\arctan\left(\frac{1}{2Kv}\right)=\frac{1}{t},$$

which is $\Delta(t,K)$. The claim then follows from Corollary 3. $\square$

Remark. By Lemma 6 and Proposition 5, the supremum of the density of μ_t is strictly smaller than that of μ, however, the estimate given by Lemma 6 is useful only if $K<1/(\pi\sqrt{t})$. On the other hand, the estimate given by $\Delta(t,K)$ is sharp, since it corresponds to the case of the uniform distribution on some interval.

Among other interesting phenomena which can be inferred from our formulas, we give the following simple example.

Proposition 6. *Let μ be symmetric, with a continuous strictly positive density on $]-\varepsilon,0[\,\cup\,]0,\varepsilon[$, satisfying*

$$\int_{\mathbb{R}}\frac{1}{x^2}p(x)dx<\infty.$$

Then there exists $t_0>0$ such that $p_t(0)=0$ and $p_t(x)>0$ for $x\in\,]-\varepsilon,\,0[\,\cup\,]0,\varepsilon[$ and $t\le t_0$.

Proof. The hypotheses on μ imply that $v_t(0)<\infty$ while $v_t(x)=\infty$ for $x\in\,]-\varepsilon,0[\,\cup\,]0,\varepsilon[$. Furthermore, by symmetry, $\psi_t(x)\le x$ for $x\le 0$ and $\psi_t(x)\ge x$ for $x\ge 0$, for all $t>0$. The conclusion follows easily from these remarks and Corollary 3. $\square$

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