

Hypothesis Tests concerning variances

April 15, 2020

1 Estimating σ^2

Just like in the last section where we considered hypothesis tests for an unknown mean, we can do the same with an unknown variance. We will once again consider the small sample size regime, it turns out in the large sample regime the theory would be very similar to the estimating means, but we won't go too much into it.

****Throughout this section we will always assume Y_i is a normal random variable with unknown mean μ and variance σ^2 .****

The random variable

$$\frac{(n-1)S^2}{\sigma^2}$$

has a χ^2 distribution with $n-1$ df, where, as usual,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

So we can use this random variable as our test statistic and we proceed as we did in the last section. The main difference will be that this random variable is not symmetric, it is always positive, so choosing our RR will be a little different.

HYPOTHESIS TEST 1.1 (Hypothesis Test for σ^2). *Let Y_1, Y_2, \dots, Y_n be a random sample from a normal distribution with unknown variance σ^2 .*

The null hypothesis is $H_0: \sigma^2 = \sigma_0^2$.

The alternative hypothesis is one of the three H_a : $\begin{cases} \sigma^2 > \sigma_0^2 \\ \sigma^2 < \sigma_0^2 \\ \sigma^2 \neq \sigma_0^2 \end{cases}$

The test statistic is $W = \frac{(n-1)S^2}{\sigma^2}$.

The rejection region is one of the three (depending on H_a) RR: $\begin{cases} W > \chi_{\alpha, (n-1)df}^2 \\ W < \chi_{1-\alpha, (n-1)df}^2 \\ W > \chi_{\alpha/2, (n-1)df}^2 \text{ or } W < \chi_{1-\alpha/2, (n-1)df}^2 \end{cases}$

Where $\chi_{\alpha, (n-1)df}^2$ is such that $\mathbb{P}(W > \chi_{\alpha, (n-1)df}^2) = \alpha$, where W has (as above) the χ^2 -distribution with $n-1$ degrees of freedom.

On a high level, sometimes a small variance is desirable, it means the objects in your data set all behave in a similar manner. Other times a large variance might be desirable, it means each time you sample you random variables you'll see different behavior. If you're sampling from a large probability space, you might like to be able to observed many of the different behaviors the random variables can take. It is important to remember the distinction between the variance that the random variables have, and the variance (or mean square error) of the estimators we studied.

Let's look at an example.

EXAMPLE 1.2. *A manufacturer of hard hats is concerned with the mean and variance of the forces the helmets transmit to their wearers. They desire the mean force transmitted to be 800 lbs. (or less) with a standard deviation less than 40 lbs. Tests were run on a random sample of $n=40$ hard hats. The sample mean and variance were found to be 825 lbs. and 2350 lbs.², respectively.*

At what level of significance does the data provide evidence that σ is greater than 40?

Solution:

We wish to test $H_0: \sigma = 40$ against $H_a: \sigma > 40$, using the test statistic $W = \frac{(n-1)S^2}{\sigma^2}$, with $n=40$.

Under H_0 , $W = \frac{39(2350)}{40^2} = 57.28$, where the 40^2 in the denominator is σ^2 under H_0 .

Since we are doing a one sided test, we now need to compute the probability that a χ^2 random variable with $39df$ is larger than 57.28:

$$\mathbb{P}(W > 57.28) = 0.0296$$

which I computed on R using the command: `1-pchisq(57.28,df=39)`.

So we can reject the null hypothesis at a level $\alpha = .0296$ or higher. So there is strong evidence that the variance of the forces transmitted is too high.

2 Comparison of Variances

In a similar manner to the comparison on the means in the last section, we can also compare the variance of the two different data sets. The main difference is: instead of considering the difference, we will consider the ratio of the variances. It is a bit more natural because variances are always positive. It turns out the ratio of two independent χ^2 random variables also has a nice description.

We consider X_1, X_2, \dots, X_m a random sample from a normal distribution with unknown variance σ_X^2 and Y_1, Y_2, \dots, Y_n a random sample from a normal distribution with unknown variance σ_Y^2 . We assume the X_i 's and independent from the Y_i 's

Since we want to consider the ratio of the variances we begin with the statistic:

$$\frac{S_X^2}{S_Y^2}$$

where as usual $S_X^2 = \frac{1}{m-1} \sum_i^m (X_i - \bar{X})^2$ and $S_Y^2 = \frac{1}{n-1} \sum_i^n (Y_i - \bar{Y})^2$.

To turn this into a pivotal quantity, we follow the end of last class's notes that and use the F distribution:

$$F = \frac{\frac{m-1}{\sigma_X^2(m-1)} S_X^2}{\frac{n-1}{\sigma_Y^2(n-1)} S_Y^2} = \frac{S_X^2 \sigma_Y^2}{S_Y^2 \sigma_X^2}$$

with $m-1$ numerator df and $n-1$ denominator df and use this as our test statistic.

We first consider the hypothesis test for $\sigma_X^2 > \sigma_Y^2$. We can then modify the test to get a test for $\sigma_X^2 < \sigma_Y^2$ and then extend this idea to get a test for $\sigma_X^2 \neq \sigma_Y^2$.

HYPOTHESIS TEST 2.1 (Hypothesis Test for $\sigma_X^2 > \sigma_Y^2$). *Let X_1, X_2, \dots, X_m be a random sample from a normal distribution with unknown variance σ_X^2 , and let Y_1, Y_2, \dots, Y_n be a random sample from a normal distribution with unknown variance σ_Y^2 . Furthermore assume the X_i 's and independent from the Y_i 's*

The null hypothesis is $H_0: \sigma_X^2 = \sigma_Y^2$.

The alternative hypothesis is $H_a: \sigma_X^2 > \sigma_Y^2$.

The test statistic is $F = \frac{S_X^2}{S_Y^2}$.

The rejection region is $RR: \frac{S_X^2}{S_Y^2} > F_{\alpha, (m-1)/(n-1)df}$

Where $F_{\alpha, (m-1)/(n-1)df}$ is such that $\mathbb{P}(F > F_{\alpha, (m-1)/(n-1)df}) = \alpha$, where F has (as above) the F -distribution with $m-1$ numerator degrees of freedom and $n-1$ denominator degrees of freedom.

Note that under H_0 , $\sigma_X^2 = \sigma_Y^2$, so they cancel when computing the rejection region.

The Hypothesis test for $H_a: \sigma_X^2 < \sigma_Y^2$ could be written similarly but it's easier to instead consider the test statistic $1/F = \frac{S_Y^2}{S_X^2}$, which is now has F with with $n-1$ numerator df and $m-1$ denominator df. So our rejection region becomes

$$RR: \frac{S_Y^2}{S_X^2} > F_{\alpha, (n-1)/(m-1)df}$$

which is still an upper tail test. If you want to write it as a lower-tail test, just take the reciprocal:

$$RR: \frac{S_X^2}{S_Y^2} < \frac{1}{F_{\alpha, (n-1)/(m-1)df}}$$

Finally the Hypothesis test for $H_a: \sigma_X^2 \neq \sigma_Y^2$ is done by the combining the last two tests with half the probability of a type I error split between the two pieces:

$$RR: \frac{S_Y^2}{S_X^2} > F_{\alpha/2, (m-1)/(n-1)df} \text{ or } \frac{S_Y^2}{S_X^2} < \frac{1}{F_{\alpha/2, (n-1)/(m-1)df}}$$

This rejection region can also we written

$$RR: \frac{S_L^2}{S_S^2} > F_{\alpha/2, (m-1)/(n-1)df}$$

where S_L^2 is the greater of S_Y^2 and S_X^2 and S_S^2 is the smaller.

Part of the reason for writing all the quantities in terms of $F_{\alpha, (m-1)/(n-1)df}$ which we define as the upper tail probability: $\mathbb{P}(F > F_{\alpha, (m-1)/(n-1)df}) = \alpha$, is so that only half of the tables need to be written.

EXAMPLE 2.2. The disinfectant properties of bleach and ethanol are to be compared. A number of samples is disinfected by each and then the number of uncontaminated samples is counted after a given time period. This experiment is done for bleach 15 times and also for ethanol 15 times the following data is recorded:

	Ethanol	Bleach
Sample size	15	15
Empirical mean	3.73	4.8
Empirical Variance	2.781	1.714

- (a) What assumptions do we need to make to perform a Hypothesis test?
 (b) At what p -value from can you conclude the true variances are different?

Solution:

- (a) We assume that the number of uncontaminated samples are independent normal random variables (or at least close to normal).
 (b) Since we are doing a two-sided test (we want to check if they are different), the rejection region will be of the form:

$$RR: \frac{S_L^2}{S_S^2} > F_{\alpha/2, (m-1)/(n-1)df}$$

Where S_L^2 is the larger variance, in this case 2.781 and S_S^2 is the smaller variance, in this case 1.714. Since

$$\mathbb{P}(F > \frac{2.781}{1.714}) = \mathbb{P}(F > \frac{2.781}{1.714}) = \mathbb{P}(F > 1.6225) = .188$$

(the R command for this is `1-pf(1.6225,df1=14,df2=14)`)

So we conclude that only at a $\alpha = 2 * .188 = .376$ level (or higher) are the variances different. We multiply by 2 because we are doing a two-sided test and assumed the numerator was the bigger empirical variance.