Relationship between Hypothesis Testing and Confidence intervals

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1 Relationship between Hypothesis Testing and Confidence intervals

The methods we're using in this chapter should feel sort of similar to what we did with confidence intervals in Chapter 8. In this section we'll make that idea more precise. We'll restrict the discussion to the large sample size case, when the distribution of the estimator is approximately normal, but similar ideas could be used for other distributions.

The main observation of this section is that $\hat{\theta}$ lies outside the rejection region if and only if θ_0 lies in the coresponding confidence interval.

Let's begin by recalling the set-up for large sample sizes in each of the problems.
For both types of problems, we begin with the test statistic $\hat{\theta}$ and standardize it:

$$\frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}}.$$

To then create a confidence intervals we can use the z-scores, to get that with confidence $(1-\alpha)$ the unknown parameter lies in the interval:

$$[\hat{\theta}-\sigma_{\hat{\theta}}z_{\alpha/2},\hat{\theta}+\sigma_{\hat{\theta}}z_{\alpha/2}].$$

Note that this doesn't give any preference to particular points in the interval, it merely says with the given probability, θ lies somewhere in the interval.
In a large sample size two-sided Hypothesis test for an unknown parameter θ we have:

- $H_0:\theta=\theta_0$.
- $H_a:\theta\neq\theta_0$.
- The test statistic is the estimator $\hat{\theta}$.
- The RR is $|\hat{\theta}-\theta_0|>\sigma_{\hat{\theta}}z_{\alpha/2}$.

and we reject H_0 if:

$$|\hat{\theta}-\theta_0|\geq \sigma_{\hat{\theta}} z_{\alpha/2}$$

or in other words we fail to reject H_0 if θ_0 lies in the interval:

$$[\hat{\theta}-\sigma_{\hat{\theta}} z_{\alpha/2},\hat{\theta}+\sigma_{\hat{\theta}} z_{\alpha/2}].$$

Which is quite similar to the confidence interval, so similar to the confidence interval, if you fail to rejection H_0, you're not claiming that H_0 is true in the sense that $\theta=\theta_0$, but instead that with the desired probability, θ lies somewhere in the complement of the rejection region. In practice, when one fails to reject the H_0, one should develop a new H_a to test.

Let's work this out in an example. It won't involve anything new, it just serves to highlight the similarities in what we are currently studying and what we did before. We'll first consider a 2-sided example.

EXAMPLE 1.1. *Two cities are trying different mitigation strategies to stop the spread of a certain contagious disease. The first city estimates a growth rate of* 1.2 *with a standard error of* .2*, the second city estimates a growth rate of* 1.35 *with a standard error of* .15*. Assume the estimators follow a normal distribution.*

(a) Give a 95% *confidence interval for the difference in the true grow rates. Is there significant evidence the cities have different growth rates?*

(b) At a level $\alpha=.05$*, test the null hypothesis that the growth rates are the same against the alternative hypothesis that the are not.*

Solution:

(a) Let θ_1 be the true growth rate of the first city, and let θ_2 be the true growth rate of the second city. By our normality assumption we have that, after standardizing:

$$\mathbb{P}(-z_{.05/2}\leq \frac{(\hat{\theta}_1-\hat{\theta}_2)-(\theta_1-\theta_2)}{\sqrt{\sigma_{\hat{\theta}_1^2}+\sigma_{\hat{\theta}_2^2}}}\leq z_{.05/2})=.05$$

Since $z_{.025}=1.96$, $(\hat{\theta}_1-\hat{\theta}_2)=1.2-1.35=-.15$, and $\sqrt{\sigma_{\hat{\theta}_1^2}+\sigma_{\hat{\theta}_2^2}}=\sqrt{.2^2+.15^2}=.25$

We have with confidence 95% $\theta_1-\theta_2$ lies in the interval $[-.15-1.96*.25,-.15+1.96*.25]=[-.64,.34]$.

Since 0 lies in this confidence interval, we can not conclude there is a differences in the true growth rates. (Note that in the next part, 0 is the value of $\theta_1-\theta_2$ under H_0.)

(b) Using the same notation we are testing $H_0:\theta_1=\theta_2$ versus $H_a:\theta_1\neq\theta_2$. Under H_0 we have,

$$\mathbb{P}(-z_{.05/2}\leq \frac{\hat{\theta}_1-\hat{\theta}_2}{\sqrt{\sigma_{\hat{\theta}_1^2}+\sigma_{\hat{\theta}_2^2}}}\leq z_{.05/2})=.05$$

or rearranging (and using the computations from the previous part), we have with probability 95%

$$(\hat{\theta}_1-\hat{\theta}_2)\in[-1.96*.25,1.96*.25]=[-.49,.49].$$

Meaning RR is $|(\hat{\theta}_1-\hat{\theta}_2)|>.49$.

Since the observed $(\hat{\theta}_1-\hat{\theta}_2)=.15$ is not in RR, we do not reject the null hypothesis.

You can repeat the above problem this a measured growth rate of 1.1 and 1.7, and see that in this case the confidence interval would not contain 0 and the test statistic would lie in RR, so both approaches would lead us to conclude the growth rates are different.

Let's do one more example with one-side confidence intervals/hypothesis tests.

EXAMPLE 1.2. *A group of researchers is trying to determine the percentage of the population that have a certain disease. Based on similar cities they suspect* 20% *of the population is infected, and they have determined that if more than* 20% *of the population is infected, critical actions are required. To determine the percentage of the population infected they test a random sample of 900 people. Let p be the true proportion that have the disease. Let $\hat{p}$ be the proportion measured with the disease.*

(a) Give a 95% *lower confidence bound for the proportion infected (as a function of $\hat{p}$). For what values of $\hat{p}$ is there evidence that $p>.2$?*

(b) At a level $\alpha=.05$, test the null hypothesis $p=.2$ against the alternative hypothesis $p>.2$.

Solution:

(a) Since we are in the large sample regime we assume $\hat{p}$ is normally distribution. We will take $\sigma_{\hat{p}}=\frac{\sqrt{.2(1-.2)}}{\sqrt{900}}=1/75\approx.0133$. So we have

$$\mathbb{P}\left(\frac{\hat{p}-p}{1/75}>z_{.05}\right)=.05$$

Rearranging and using that $z_{.05}=1.65$.

$$\mathbb{P}\left(\hat{p}-\frac{1.65}{75}>p\right)=.05$$

We have with probability .95

$$\hat{p}-\frac{1.65}{75}<p.$$

In other words, the 95% lower confidence interval is $[\hat{p}-\frac{1.65}{75},\infty)$.

So if $\hat{p}>.2+\frac{1.65}{75}$, we can conclude with 95% confidence that p is greater than .2.

(b) We are testing $H_0:p=.2$ against $H_a:p>.2$ with the test statistic $\hat{p}$, RR is

$$\frac{\hat{p}-.2}{1/75}>1.65$$

where we have used the under H_0, $p=.2$, $\sigma_{\hat{p}}=1/75$, $z_{.05}=1.65$ (see part a)). Which rearranges to

$$\hat{p}>.2+\frac{1.65}{75}$$

If we compare the two solutions we see we reject H_0 exactly when .2 is not in the lower confidence interval.