Calculating Type II error probabilities and finding the Sample size for Z-tests

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1 Calculating type II errors

In the previous section we just considered Type I errors. These were more straightforward to compute because H_0 involves only a single value for the unknown parameter. On the other hand, H_a involves a range of parameters so working with Type II errors can be a bit more involved. For this chapter, we will restrict H_a to also being a single parameter value. Later we will see how to handle the case that H_a involves an interval.

The types of problems in this chapter can be considered as classification problems; we're assuming exactly one of two outcomes will occur (H_0 or H_a) and from our data we want to select the correct one with the highest probability.

Let's look at a simple example:

EXAMPLE 1.1. *Let $Y_1, Y_2, \ldots, Y_{40}$ be a random sample sample with unknown mean θ and standard deviation 2. Suppose we want to test the null hypothesis H_0, $\theta_0 = 130$, against the alternative hypothesis H_a, $\theta_a = 128$, by using the test statistic $\frac{1}{40}\sum_{i=1}^{40} Y_i$.*

(a) Determine the RR if we desire $\alpha = .05$.

(b) What is the corresponding β for this rejection region?

Solution:

(a) The first is similar to the last section, since $\theta_a < \theta_0$ we desire a RR of the form

$$\mathbb{P}(\overline{Y} < k) = .05$$

Standardizing as in the last section (so assuming H_0 holds) leads to

$$\mathbb{P}\left(\frac{\overline{Y} - 130}{2/\sqrt{40}} < \frac{k - 130}{2/\sqrt{40}}\right) = .05$$

(We used the given $\sigma = 2$ and $n = 40$.) Then since we are in the large sample region, we assume $\overline{Y}$ is approximately normal so we have

$$\frac{130 - k}{2/\sqrt{40}} = z_{.05} = 1.645$$

(Note we multiplied by -1 because we want k less than 130.) Solving gives

$$k = 130 - 1.645 * 2/\sqrt{40} = 129.48$$

So RR is $\overline{Y} < 129.48$.

(b) To determine β we now assume $\theta = 128$ and calculate the probability $\overline{Y} > 129.48$. Under H_a the sample size and variance are the same as H_0, so we just need to change the mean.

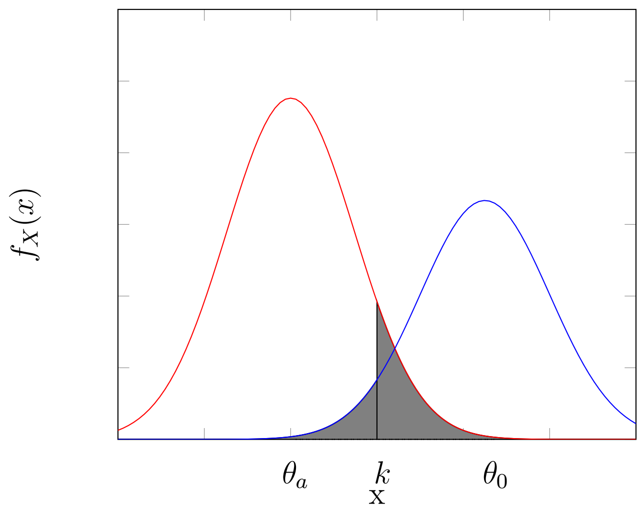
$$\mathbb{P}\left(\frac{\overline{Y}-128}{2/\sqrt{40}} > \frac{129.48-128}{2/\sqrt{40}}\right)$$

Note we now use the $>$ because we're computing the probability we don't reject H_0. Simplifying gives:

$$\mathbb{P}\left(\frac{\overline{Y}-128}{2/\sqrt{40}} > 4.67\right) \approx .000003$$

The choice of $n=40$ and $\alpha=.05$, in the above example, led to a very small β. In some sense this was not so efficient, we could have used a smaller sample size n, and kept α the same and had a larger, but still small β.

In general you should have the following picture in mind:



The blue curve on the right denotes the distribution of the test statistic H_0 when is true and the red curve on the left denotes the distribution of the test statistic when H_a is true. The means of each distribution, θ_a and θ_0, are marked. In between θ_a and θ_0 is k, the boundary of RR, to the left of k we reject H_0.

The area of the two shaded regions represents the probability of an error. The left area is α because if the test statistic lands to the left of k we reject H_0, and this area represents the probability of the test statistic landing to the left of k when H_0 is true. Similarly, the right area is β.

2 Determining Sample Size

If we have a desired α and β then we can determine a minimum n achieve this. For now let's assume the standard deviation under H_0 and H_a is the same, but the argument can be modified for more general cases. Once again, we'll assume $\mu_a < \mu_0$. From the previous analysis we know

$$\frac{k-\mu_0}{\sigma/\sqrt{n}} = -z_\alpha \text{ and } \frac{k-\mu_a}{\sigma/\sqrt{n}} = z_\beta$$

Solving for k in each equation gives:

$$k = \mu_0 - \frac{z_\alpha \sigma}{\sqrt{n}} \text{ and } k = \mu_a + \frac{z_\beta \sigma}{\sqrt{n}}$$

Setting right sides equal and solving for n leads to

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_a)^2}$$

being the smallest n that achieves the desired values for α and β. Note that as $\mu_0 - \mu_a$ become smaller, n will increase. Likewise if σ increases, or α or β decrease we will also need a larger n.