Moment generating Function

March 16, 2020

1 Moments

Just like in the discrete case we can also consider the moments of continuous random variables. Recall if X is a discrete random variable with pmf $p_X(x)$ then its k^{th} moment (where k is a positive integer) is denoted m'_k and equals

$$m'_k = \mathbb{E}[X^k] = \sum_x x^k p_X(x)$$

In the continuous case, we follow our general rule that pmf $\rightarrow$ pdf and $\sum \rightarrow \int$, and see if Y is a continuous random variable, with pdf $f_Y(y)$, then

$$m'_k = \mathbb{E}[Y^k] = \int y^k f_Y(y)dy.$$

You should think of higher moments, as giving information about how likely it is the random variable takes a large value. For larger k the y^k is much bigger for larger y than smaller. Inside the integral we multiply this weight y^k by $f_Y(y)$. As $f_Y(y)$ tells us the probability Y is near y, the factor y^k is applying much more weight to this term. The integral is then this weighted sum of the probability density.

EXAMPLE 1.1. *Let Y be continuous random variable with pdf $f_Y(y) = y/2$ for $0 \le y \le 2$. Compute the k^{th} moment of Y.*

Solution:

$$\mathbb{E}[Y^k] = \int_0^2 y^k \left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^{k+1}}{2} dy = \left.\frac{y^{k+2}}{2(k+2)}\right|_0^2 = \frac{2^{k+1}}{k+2}$$

2 Moment Generating Function

Just like in the discrete case we can also define the Moment Generating Function for continuous random variables. Recall if X is a discrete random variable with pmf $p_X(x)$ then its MGF is

$$m_X(t) = \mathbb{E}[e^{tX}] = \sum_x e^{tx} p_X(x).$$

In the continuous case, we follow our general rule that pmf $\rightarrow$ pdf and $\sum \rightarrow \int$, and get the following definition:

DEFINITION 2.1. *Let Y be a continuous random variable, with pdf $f_Y(y)$. The moment generating function (MGF) of Y is*

$$m_Y(t) = \mathbb{E}[e^{tY}] = \int e^{ty} f_Y(y) dy$$

Just as in the discrete case we can use the MGF to compute the moments of Y (see examples below). For nice classes of random variables (which we will generally restrict ourselves to) knowing the moment generating function is equivalent to knowing the pdf. In some applications the MGF is nicer to work with, we'll use this fact more next Chapter. For now we'll focus on computing MGFs and computing moments from the MGF.

EXAMPLE 2.2. *Let X be exponential random variable with mean 3. Compute the MGF of X.*

Solution: *The pdf of X is*

$$f_X(x) = \frac{1}{3} e^{-x/3} \text{ for } x > 0$$

and $f_X(x) = 0$ otherwise.

$$\begin{aligned} m_X(t) = \mathbb{E}[e^{tX}] &= \int_0^\infty e^{tx} \frac{1}{3} e^{-x/3} dx = \frac{1}{3} \int_0^\infty e^{x(t-1/3)} dx = \frac{1}{3(t-1/3)} e^{x(t-1/3)} \Big|_0^\infty \\ &= \frac{1}{3(t-1/3)} \left(\lim_{b \to \infty} e^{b(t-1/3)} - e^0 \right) = \frac{1}{1-3t} \end{aligned}$$

if $t < 1/3$. Note that if $t \geq 1/3$ then the limit above would diverge, when $t < 1/3$ the limit converges to 0. This is not an important restriction, because we are mostly interested in the MGF for t near 0.

Note, that there wasn't anything special about choosing the mean to be 3. The same computation above shows if X is an exponential random variable with mean β. Then it's MGF is

$$m_X(t) = \frac{1}{1-\beta t} \text{ for } t < \beta^{-1}$$

3 Properties of MGF

- Let X be a random variable with MGF $m_X(t)$ then for any real numbers a,b the MGF of $aX+b$ is $m_{aX+b}(t) = e^{bt} m_X(at)$. Proof:

  $$m_{aX+b}(t) = \mathbb{E}[e^{(aX+b)t}] = e^{bt} \mathbb{E}[e^{X(at)}] = e^{bt} m_X(at)$$

  We have used properties of the exponential function to separate the a and b terms. We pulled the e^{bt} term out because we can pull out any multiplicative factors, that don't depend on X, then finally observe $\mathbb{E}[e^{X(at)}]$ is simply the MGF of X evaluated at at.

- As before in the discrete case, for any r.v. X, $m_X(0) = \mathbb{E}[X^0] = \mathbb{E}[1] = 1$.

- There are other properties a MGF must satisfy, but we won't discuss them further.

4 Moment Generating Function, another example

EXAMPLE 4.1. *Let Y be a normal random variable with mean μ and variance σ^2. Compute the MGF of Y.*

Solution: *We will first compute an easier case and compute the MGF of a standard normal random variable. We can then use our formula $m_{aX+b}(t)=e^{bt}m_X(at)$, with $a=\mu$ and $b=\sigma$ to get the general case.*

Let Z be a standard normal random variable (so it has mean 0 and variance 1).

The pdf of Z is

$$f_z(z)=\frac{1}{\sqrt{2\pi}\sigma}e^{-z^2/2} \text{ for all } t$$

$$m_Z(t)=\mathbb{E}[e^{tZ}]=\int_{-\infty}^{\infty}e^{tz}\frac{1}{\sqrt{2\pi}\sigma}e^{-z^2/2}dz$$

The trick to computing this integral is to relate it to the integral of just the pdf of a normal random variable, because we were previously able to compute that integral. To achieve this we multiply the two exponential terms, this will lead to a factor of:

$$tz-z^2/2$$

in the exponent. We then complete the square to get:

$$-1/2\left(z^2-2tz+t^2\right)+t^2/2=-1/2(z-t)^2+t^2/2$$

This computation shows that

$$e^{tz}e^{-z^2/2}=e^{-1/2(z-t)^2+t^2/2}=e^{-1/2(z-t)^2}e^{t^2/2}$$

We now substitute the final term back into the integral, so:

$$m_Z(t)=\mathbb{E}[e^{tZ}]=e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-1/2(z-t)^2}dz$$

But the integral is just 1, because we are actually integrating the pdf of a normal random variable with mean t and variance 1. So we conclude

$$m_Z(t)=e^{t^2/2}$$

From the formula above we have for general normal random variables

$$m_Y(t)=e^{\frac{\sigma^2t^2}{2}+\mu t}$$

5 Computing moments of a random variable from the MGF

As in the discrete case for any positive integer k:

$$\mathbb{E}[X^k] = \frac{d^k}{dt^k} m_X(t)\bigg|_{t=0}.$$

The proof is basically the same as the discrete case, so let's look at an example.

EXAMPLE 5.1. *Let Z be standard normal random variable with mean* 0 *and variance* 1*. Use the MGF to compute the* 4^{th} *moment of* Z*.*

Solution:

$$\mathbb{E}[Z^4] = \frac{d^4}{dt^4} m_Z(t)\bigg|_{t=0} = \frac{d^4}{dt^4} e^{t^2/2}\bigg|_{t=0}$$

where we have just substituted the MGF that we just computed. Now we do the derivatives:

$$\frac{d}{dt} e^{t^2/2} = te^{t^2/2}$$
$$\frac{d^2}{dt^2} e^{t^2/2} = e^{t^2/2} + t^2 e^{t^2/2}$$
$$\frac{d^3}{dt^3} e^{t^2/2} = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} = 3te^{t^2/2} + t^3 e^{t^2/2}$$
$$\frac{d^4}{dt^4} e^{t^2/2} = 3e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^2 e^{t^2/2} + t^4 e^{t^2/2}$$

When we evaluate at $t=0$, the last 3 terms are all zero and only the first term remains:

$$\frac{d^4}{dt^4} e^{t^2/2}\bigg|_0 = 3e^0 + 0 = 3$$

So the 4th moment of a standard normal random variable is 3.

While this is not the easiest computation we've done, computing derivatives can often be more straightforward than computing integrals. The fourth moment could have also been computed direction from the definition of moments and computing the integral of z^4 times the pdf. Which technique to use sort of a matter of taste, and somewhat problem dependent.

6 Identifying the distribution of a random variable from its MGF

Given a pdf, the MGF is computed from the integral in the definition. Going from a MGF back to the pdf is generally much more difficult. For this class, we will generally do this by "pattern matching". Above we have computed MGF for several classes of random variables, the book does several more and in the back of the book there is a table which includes the MGF of several common random variables. So given a MGF, you simply want to match it to a known one, and determine what the correct parameters are. Below are several examples:

EXAMPLE 6.1. *Determine the distribution of random variables with the following MGFs:*

1. $m_X(t) = \frac{e^{t3} - e^{-t}}{t(4)}$
2. $m_Y(t) = \frac{e^t}{4 - (4-1)e^t}$
3. $m_Z(t) = e^t(1-2t)^{-1}$

Solution:

1. *From the table we see* X *is a uniform random variable on the interval* $[-1,3]$. *That means, it's pdf is:*

$$f_X(x) = \begin{cases} \frac{1}{4}, & \text{if } -1 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

2. *This one isn't exactly on the table, but look like geometric, so should try to manipulate it to make it look like a geometric, indeed we would like the first term on the bottom to be a one, so we multiply top and bottom by* $1/4$ *to get* $m_Y(t) = \frac{1/4e^t}{1-(1-1/4)e^t}$, *and we see* Y *is a geometric random random variable with* $p = 1/4$. *That means, it's pmf (it's discrete so we use pmf) is:*

$$p_Y(k) = \begin{cases} \left(\frac{3}{4}\right)^{k-1}\frac{1}{4}, & \text{for } k \text{ an integer} \\ 0, & \text{otherwise.} \end{cases}$$

3. *This one is a bit tricky, it's not on the table, but it's close, if the* e^t *factor wasn't there, it would be an exponential r.v. with mean* 2. *But then remember that adding a constant b to a r.v. multiplies its MGF by* e^{bt}. *So we see* Z *is equal to 1 plus an exponential r.v. with mean* 2. *That means, it's pdf is:*

$$f_X(x) = \begin{cases} \frac{1}{2}e^{\frac{x}{2}-1}, & \text{if } x \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$