Quiz 2

1. Evaluate the integral:

$$\int \frac{2^x}{2^x + 1}\, dx .$$

2. Find the absolute minimum of the function $g(x) = \dfrac{e^x}{x}$, $x > 0$.

3. Differentiate the function:

$$g(x) = \ln\left(x\sqrt{x^2 - 1} \right).$$

Answers

1. Q: Evaluate: $\int \frac{2^x}{2^x+1}\,dx$

A: Let $u = 2^x + 1$

then $du = (\ln 2) \cdot 2^x \, dx$

so $\frac{du}{\ln 2} = 2^x \, dx$.

Substituting, we get

$$\int \frac{2^x}{2^x+1}\,dx = \int \frac{1}{u} \cdot \frac{du}{\ln 2}$$

$$= \frac{1}{\ln 2} \ln u + C$$

$$= \frac{1}{\ln 2} \ln(2^x+1) + C$$

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② Q: Find absolute minimum of $g(x) = \frac{e^x}{x}$, $x > 0$.

A: We'll need to examine the critical points of $g(x)$ and also $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to \infty} g(x)$.

(i) Critical points occur when $g'(x) = 0$ or $g'(x)$ is not defined.

$$g'(x) = \frac{xe^x - e^x(1)}{x^2} = e^x\left(1 - \frac{1}{x}\right).$$

Now, for $x > 0$, $e^x > 0$, so $g'(x) = 0$ only when $1 - \frac{1}{x} = 0$, which happens when $x = 1$. So $x = 1$ is a critical value. And because $g'(x)$ is defined for all $x > 0$, $x = 1$ is the only critical value.

We now look at the sign of $g'(x)$ near $x = 1$:

	$x<1$	$x>1$
e^x	+ + + + + +	+ + + + + +
$1 - \frac{1}{x}$	− − − −	+ + + + + +
$g'(x) = e^x\left(1 - \frac{1}{x}\right)$	− − − −	+ + + + + + +

$x = 1$

This tells us there's a local min of $g(x)$ at $x=1$.

(ii) Checking limits:

$$\lim_{x \to 0^+} \frac{e^x}{x} = \frac{\lim_{x \to 0^+} e^x}{\lim_{x \to 0^+} x}$$

$$= \frac{1}{\lim_{x \to 0^+} x} = \infty$$

And

$\lim_{x \to \infty} \frac{e^x}{x} = \infty$ because e^x is an exponential function with base $e \approx 2.7 > 1$, and such functions grow faster than any polynomial, which x is.

So we have

$$\lim_{x \to 0^+} g(x) = \lim_{x \to \infty} g(x) = \infty .$$

Hence minimum at $x=1$ is a global minimum on $\{x \mid x > 0\}$. //

③ Q: Differentiate $g(x) = \ln\left(x\sqrt{x^2-1}\right)$.

A: $$g'(x) = \frac{1}{x\sqrt{x^2-1}} \cdot \frac{d}{dx}\left[\underbrace{x}_{f}\,\underbrace{\sqrt{x^2-1}}_{h}\right]$$

$$= \frac{1}{x\sqrt{x^2-1}} \cdot \left[\underbrace{x}_{f} \cdot \underbrace{\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}} \cdot 2x}_{h'} + \underbrace{\sqrt{x^2-1}}_{h}\,\underbrace{(1)}_{f'}\right]$$

$$= \frac{x}{x^2-1} + \frac{1}{x}$$

$$= \frac{x^2 + x^2 - 1}{x\left(x^2-1\right)}$$

$$= \frac{2x^2-1}{x\left(x^2-1\right)}$$

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