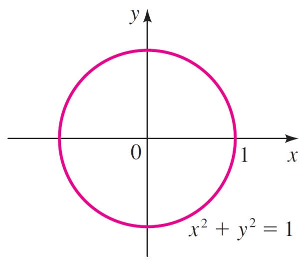
Section 5.1 The Unit Circle

The Unit Circle

The Unit Circle

The **unit circle** is the circle of radius 1 centered at the origin in the xy-plane. Its equation is

$$x^2 + y^2 = 1$$



EXAMPLE: Show that the point $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

Solution: We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

EXAMPLE: The point $(\sqrt{3}/2, y)$ is on the unit circle in Quadrant IV. Find its y-coordinate.

Solution: Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \pm\frac{1}{2}$$

Since the point is in Quadrant IV, its y-coordinate must be negative, so $y = -\frac{1}{2}$.

EXAMPLE:

(a) Is the point $\left(\frac{\sqrt{24}}{7}, \frac{\sqrt{26}}{7}\right)$ on the unit circle?

(b) The point $(\sqrt{35}/6, y)$ is on the unit circle in Quadrant I. Find its y-coordinate.

EXAMPLE:

(a) Is the point $\left(\dfrac{\sqrt{24}}{7}, \dfrac{\sqrt{26}}{7}\right)$ on the unit circle?

(b) The point $(\sqrt{35}/6, y)$ is on the unit circle in Quadrant I. Find its y-coordinate.

Solution:

(a) Since

$$\left(\frac{\sqrt{24}}{7}\right)^2 + \left(\frac{\sqrt{26}}{7}\right)^2 = \frac{24}{49} + \frac{26}{49} = \frac{50}{49} \neq 1$$

P is not on the unit circle.

(b) Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{35}}{6}\right)^2 + y^2 = 1$$

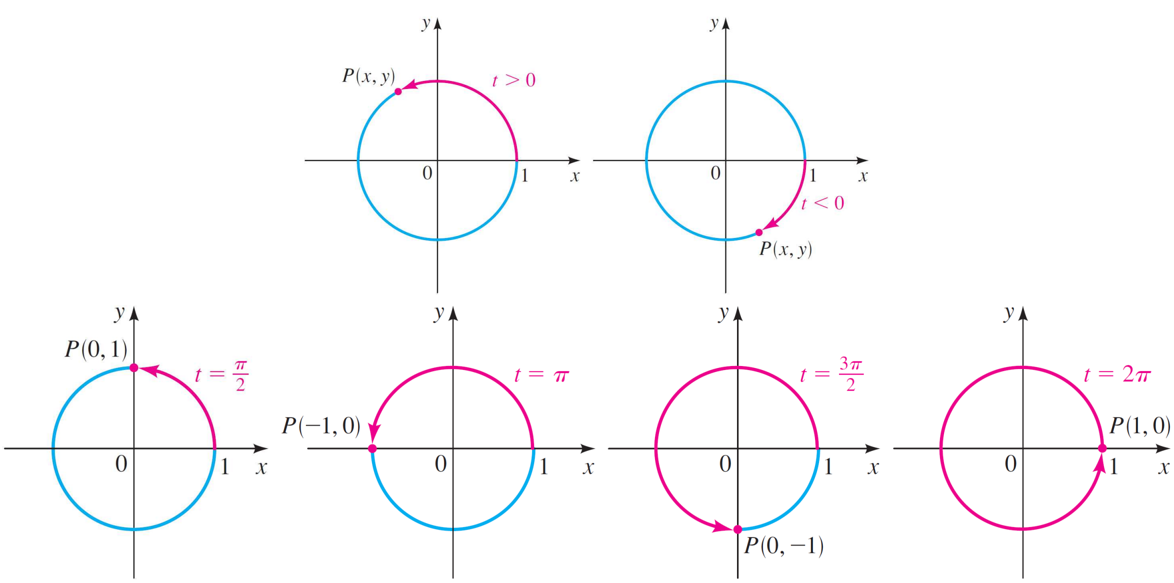
$$y^2 = 1 - \left(\frac{\sqrt{35}}{6}\right)^2 = 1 - \frac{35}{36} = \frac{1}{36}$$

$$y = \pm\frac{1}{6}$$

Since the point is in Quadrant I, its y-coordinate must be positive, so $y = \dfrac{1}{6}$.

Terminal Points on the Unit Circle

Suppose t is a real number. Let's mark off a distance t along the unit circle, starting at the point $(1, 0)$ and moving in a counterclockwise direction if t is positive or in a clockwise direction if t is negative. In this way we arrive at a point $P(x, y)$ on the unit circle. The point $P(x, y)$ obtained in this way is called the **terminal point** determined by the real number t.



EXAMPLE: Find the terminal point on the unit circle determined by each real number t.

(a) $t = 3\pi$ (b) $t = -\pi$ (c) $t = -\dfrac{\pi}{2}$ (d) $t = \dfrac{3\pi}{2}$

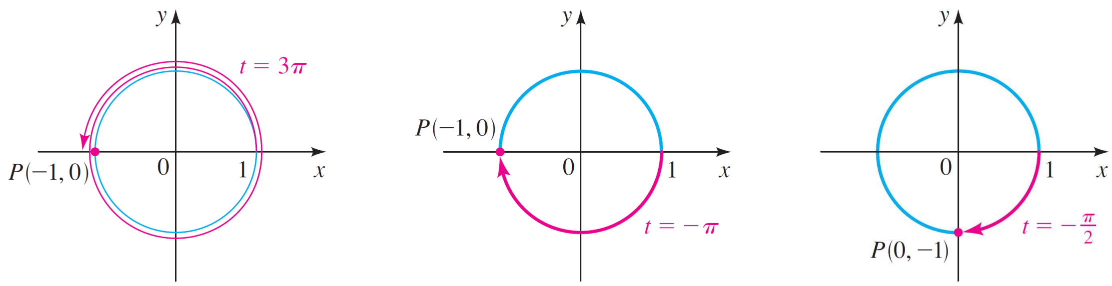
Solution:

(a) The terminal point determined by 3π is $(-1, 0)$.

(b) The terminal point determined by $-\pi$ is $(-1, 0)$.

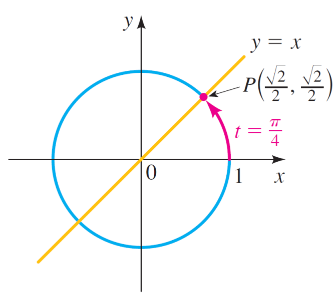
(c) The terminal point determined by $-\dfrac{\pi}{2}$ is $(0, -1)$.

(d) The terminal point determined by $\dfrac{3\pi}{2}$ is $(0, -1)$.



REMARK: Notice that different values of t can determine the same terminal point.

The terminal point $P(x, y)$ determined by $t = \pi/4$ is the same distance from $(1, 0)$ as from $(0, 1)$ along the unit circle.



Since the unit circle is symmetric with respect to the line $y = x$, it follows that P lies on the line $y = x$. So P is the point of intersection (in the first quadrant) of the circle $x^2 + y^2 = 1$ and the line $y = x$. Substituting x for y in the equation of the circle, we get

$$x^2 + y^2 = 1$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

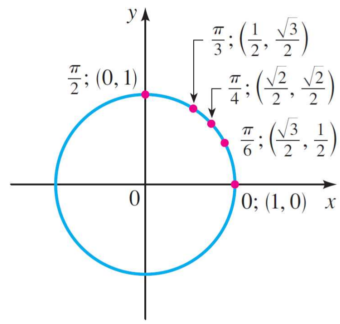
$$x = \pm\frac{1}{\sqrt{2}}$$

Since P is in the first quadrant, $x = 1/\sqrt{2}$ and since $y = x$, we have $y = 1/\sqrt{2}$ also. Thus, the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$. The Table and Figure below give the terminal points for some special values of t.

t	Terminal point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$



EXAMPLE: Find the terminal point determined by each given real number t.

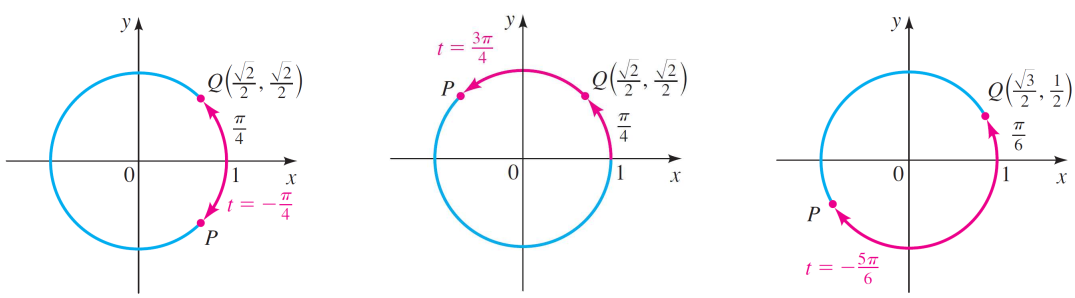
(a) $t = -\dfrac{\pi}{4}$ (b) $t = \dfrac{3\pi}{4}$ (c) $t = -\dfrac{5\pi}{6}$

Solution:

(a) The terminal point is $P\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

(b) The terminal point is $P\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

(c) The terminal point is $P\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.



EXAMPLE: Find the terminal point determined by each given real number t.

(a) $t = \dfrac{5\pi}{4}$ (b) $t = -\dfrac{\pi}{6}$

EXAMPLE: Find the terminal point determined by each given real number t.

(a) $t = \frac{5\pi}{4}$ (b) $t = -\frac{\pi}{6}$

Solution:

(a) The terminal point is $P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. (b) The terminal point is $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

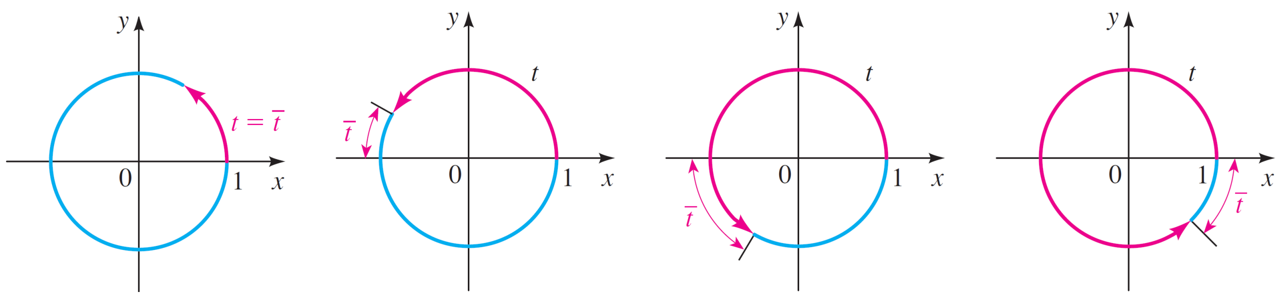
The Reference Number

From the Examples above we see that to find a terminal point in any quadrant we need only know the "corresponding" terminal point in the first quadrant. We use the idea of the *reference number* to help us find terminal points.

Reference Number

Let t be a real number. The **reference number** $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

The Figures below show that to find the reference number $\bar{t}$ it's helpful to know the quadrant in which the terminal point determined by t lies. If the terminal point lies in quadrants I or IV, where x is positive, we find $\bar{t}$ by moving along the circle to the *positive* x-axis. If it lies in quadrants II or III, where x is negative, we find $\bar{t}$ by moving along the circle to the *negative* x-axis. **The reference number $\bar{t}$ is always between 0 and $\pi/2$: $0 \le \bar{t} \le \pi/2$.**

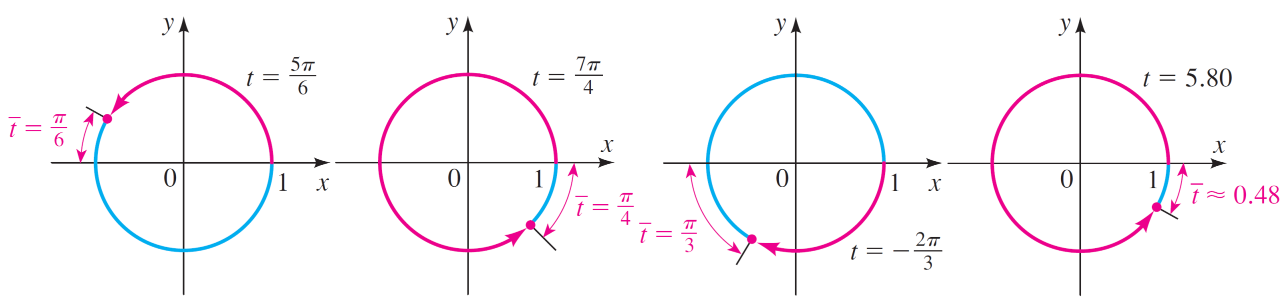


EXAMPLE: Find the reference number for each value of t.

(a) $t = \frac{5\pi}{6}$ (b) $t = \frac{7\pi}{4}$ (c) $t = -\frac{2\pi}{3}$ (d) $t = 5.80$

Solution: From the Figures below we find the reference numbers as follows.

(a) $\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$ (b) $\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$ (c) $\bar{t} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (d) $\bar{t} = 2\pi - 5.80 \approx 0.48$



EXAMPLE: Find the reference number $\bar{t}$ for $t = \frac{17\pi}{6}$.

EXAMPLE: Find the reference number $\bar{t}$ for $t = \dfrac{17\pi}{6}$.

Solution: We have

$$\bar{t} = 3\pi - \frac{17\pi}{6} = \frac{\pi}{6} \qquad \text{or} \qquad \frac{17\pi}{6} = \frac{18\pi - \pi}{6} = \frac{18\pi}{6} - \frac{\pi}{6} = 3\pi - \underbrace{\frac{\pi}{6}}_{\bar{t}}$$

Using Reference Numbers to Find Terminal Points

To find the terminal point P determined by any value of t, we use the following steps:

1. Find the reference number $\bar{t}$.
2. Find the terminal point $Q(a, b)$ determined by $\bar{t}$.
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

t	Terminal point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$

EXAMPLE: Find the terminal point determined by each given real number t.

(a) $t = \dfrac{5\pi}{6}$ (b) $t = \dfrac{7\pi}{4}$ (c) $t = -\dfrac{2\pi}{3}$

Solution: The reference numbers associated with these values of t were found in the Example on page 5.

(a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $(\sqrt{3}/2, 1/2)$ from the Table above. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

(b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point $(\sqrt{2}/2, \sqrt{2}/2)$ from the Table above. Since the terminal point determined by t is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus, the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

(c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $(1/2, \sqrt{3}/2)$ from the Table above. Since the terminal point determined by t is in Quadrant III, its coordinates are both negative. Thus, the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

EXAMPLE: Find the terminal point determined by each given real number t.

(a) $t = -\dfrac{7\pi}{6}$ (b) $t = -\dfrac{4\pi}{3}$

EXAMPLE: Find the terminal point determined by each given real number t.

(a) $t = -\dfrac{7\pi}{6}$ (b) $t = -\dfrac{4\pi}{3}$

Solution:

(a) The reference number is

$$\bar{t} = \frac{7\pi}{6} - \pi = \frac{\pi}{6} \qquad \text{or} \qquad \frac{7\pi}{6} = \frac{6\pi + \pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \pi + \underbrace{\frac{\pi}{6}}_{\bar{t}}$$

t	Terminal point determined by t
0	(1,0)
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	(0,1)

which determines the terminal point $(\sqrt{3}/2, 1/2)$ from the Table on the right. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

(b) The reference number is

$$\bar{t} = \frac{4\pi}{3} - \pi = \frac{\pi}{3} \qquad \text{or} \qquad \frac{4\pi}{3} = \frac{3\pi + \pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \pi + \underbrace{\frac{\pi}{3}}_{\bar{t}}$$

which determines the terminal point $(1/2, \sqrt{3}/2)$ from the Table above. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

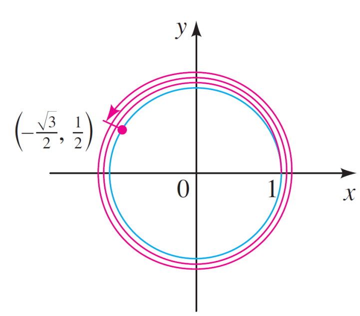
EXAMPLE: Find the terminal point determined by $t = \dfrac{29\pi}{6}$.

Solution: The reference number is

$$t = 5\pi - \frac{29\pi}{6} = \frac{\pi}{6} \qquad \text{or} \qquad \frac{29\pi}{6} = \frac{30\pi - \pi}{6} = \frac{30\pi}{6} - \frac{\pi}{6} = 5\pi - \underbrace{\frac{\pi}{6}}_{\bar{t}}$$

which determines the terminal point $(\sqrt{3}/2, 1/2)$ from the Table above. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$



EXAMPLE: Find the terminal point determined by $t = \dfrac{55\pi}{6}$.

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t	Terminal point determined by t
0	$(1, 0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0, 1)$

Solution: The reference number is

$$t = \frac{55\pi}{6} - 9\pi = \frac{\pi}{6} \qquad \text{or} \qquad \frac{55\pi}{6} = \frac{54\pi + \pi}{6} = \frac{54\pi}{6} + \frac{\pi}{6} = 9\pi + \underbrace{\frac{\pi}{6}}_{\bar{t}}$$

which determines the terminal point $(\sqrt{3}/2, 1/2)$ from the Table on the right. Since the terminal point determined by t is in Quadrant III, its coordinates are both negative. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

EXAMPLE: The Table below contains some values of t, their reference numbers, and their terminal points.

t	$\bar{t}$	Terminal Point
$-\dfrac{33\pi}{2}$	$\dfrac{\pi}{2}$	$(0, -1)$
$\dfrac{55\pi}{4}$	$\dfrac{\pi}{4}$	$\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$
8π	0	$(1, 0)$
$\dfrac{16\pi}{3}$	$\dfrac{\pi}{3}$	$\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$
10	$10 - 3\pi \approx 0.57522204$	$(-0.839, -0.544)$