6 Limits

Finite Limits

The subject of calculus revolves around the idea of a *limit* Mathematicians have precisely defined the term "limit," but we will not be so formal in this course. However, it is important that you have an accurate understanding of the concept. In order to do this, you must first understand the idea of *arbitrarily close.* An illustration, albeit rather contrived, should help with this.

Suppose you work for a strange company and it is your job to draw squares. Fortunately, you have a machine that draws squares of any size; all you have to do is tell the machine the length of the side desired. On Monday, your boss comes to you and says that he needs a square with an area of 25 square inches. Now you know that $A = s^2$ (area = side × side), so you go to set your machine for a side length of 5 inches. Unfortunately, you find that there is a malfunction in your machine, and the only side length that it isn't handling correctly is a length of 5 inches. You report this to your boss.

"Can't you just use a different length?"

"No, sir. To have a square with an area of exactly 25 in^2, I must use a side length of exactly 5 inches." (Your expertise in the matter is why you get paid the big bucks.)

"Well, maybe I could get by with a square that is little bit bigger...but I want an area less than 26 in^2. Can you do that?"

" Yes, sir. If I make the length of the side 5.05 inches, then I could give you a square with area 25.5025 in^2."

"Well, that would be OK, I guess, but maybe smaller would be better. Can you get me a square with area less than, say, 25.2 in^2?"

"Yes. I could make the length of a side 5.01 inches. That would give you an area of 25.1001 in^2."

"Well, I'm satisfied with that, but I'm not sure how my manager will react. Could you make a square with area less than 25.0001 in^2?"

"Certainly. Any side with length less than 5.00001 inches will work."

"Great! Great! You are a marvel! Oh, but wait...the CEO is coming today to check out this project. Just how close *can* you get to a square of area 25 in^2? What shall I tell her? What is the very best you can do?"

"Sir, you can tell the chief that although I cannot make a square with area exactly 25 in^2, I can make a square that has an area as close to 25 in^2 as she wishes. The difference in area between my square and a square of area 25 in^2 can be just as small as she wants. All I need to do is make sure that the length of the side is sufficiently small."

There are two concepts to be gleaned from the previous illustration. One is the idea of being *arbitrarily close.* The other is the idea of a *limit.*

In the process of making the side of the square closer and closer to 5, you make the area closer and closer to 25. However, having the area simply get closer and closer to 25 isn't enough for *arbitrarily* close.[13] Arbitrary closeness requires that no matter how small (or arbitrary) a positive difference you choose, you can obtain that difference and less. A *limit* is the entity to which you can become arbitrarily close. In this case, the unattainable area of 25 in^2 is the limit.

Let's apply these concepts to functions.

We say *the limit of $f(x)$ as x approaches the number a is L* if $f(x)$ gets arbitrarily close to L as x gets closer and closer to a. When this is so, we write $\lim_{x \to a} f(x) = L$.

In this definition, L is a real number, and a may or may not be in the domain of f.

Example 6.1. $\lim_{x \to 3}(2x+4) = 10$. Here, $f(x) = 2x+4$, $a = 3$ and $L = 10$.

We are claiming that 10 is the limit of $(2x+4)$ as x gets close to the number 3 because 10 is the number to which $(2x+4)$ becomes arbitrarily close as x gets close to 3. No matter how small you want the difference between $(2x+4)$ and 10 to be, you can achieve that difference by making x sufficiently close to 3. For instance, if you want the difference between $(2x+4)$ and 10 to be less than .01, you only have to make sure that your x is within .005 of the number 3. Check it out.

There were a lot of words in Example 6.1 about differences and closeness. Let's look at the example again. If x values get closer and closer to 3 what values does $(2x+4)$ take on? The following table shows some values. Notice that x values could be less than 3 or greater than 3 as they get closer to 3.

$x < 3$	$2x+4$	$x > 3$	$2x+4$
2	8	4	12
2.5	9	3.5	11
2.9	9.8	3.1	10.2
2.99	9.98	3.01	10.02
2.99999	9.99998	3.00001	10.00002

Can you see from the table that $2x+4$ can get arbitrarily close to 10...you need only get x sufficiently close to 3?

In Example 6.1 it is true that $f(3) = 2(3)+4 = 10$. It cannot be overstated that this is irrelevant to the limit. When we write: $\lim_{x \to a} f(x) = L$, we do not consider the actual value of the function at $x = a$. We are making a statement that says that the y values of the function are getting arbitrarily close to the number L as the x values approach a. We are saying nothing about $f(a)$.

[13] The numbers in the pattern 25.21, 25.201, 25.2001, 25.20001, etc. are getting closer and closer to the number 25. But they are not getting arbitrarily close. They are all maintaining a difference of of at least 0.2. In point of fact, this sequence of numbers is getting arbitrarily close to 25.2.

Example 6.2. $g(x) = \begin{cases} 2x+4, & x < 3 \\ 2x+4, & x > 3 \end{cases}$ $\qquad h(x) = \begin{cases} 2x+4, & x < 3 \\ 5, & x = 3 \\ 2x+4, & x > 3 \end{cases}$

$g(3)$ does not exist. $h(3) = 5$. However $\lim_{x \to 3} g(x) = 10$ and $\lim_{x \to 3} h(x) = 10$ because the limit as $x \to 3$ is not concerned about the existence or value of the function *at* $x = 3$.

You can see, then, that finding a limit as $x \to a$ is not a matter of finding $f(a)$. This is emphasized again in the following, more complicated example.

Example 6.3. $\lim_{x \to 1} \left(\frac{x^2 - 2x + 1}{x - 1} \right) = 0.$

Certainly this limit was not found by evaluating the function at $x = 1$. The function is not defined at $x = 1$.

Think about this one: First note that $x^2 - 2x + 1 = (x-1)(x-1)$. <u>When $x \neq 1$</u> we can divide above and below to get:

$$\frac{x^2 - 2x + 1}{x - 1} = x - 1.$$

By the definition of limit we are not interested in what happens when x is 1 but rather in what the value of $\frac{x^2 - 2x + 1}{x - 1}$ is as x approaches 1. And as x approaches 1 the number $x - 1$ approaches 0. Hence $\lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} = \lim_{x \to 1} (x - 1) = 0$.

Example 6.4.

$$f(x) = \begin{cases} 1 & \text{if } x \le 5 \\ -2 & \text{if } x > 5 \end{cases}.$$

Here $\lim_{x \to 5} f(x)$ does not exist. If $x \to 5$ using x values decreasing to 5, $f(x) \to -2$. But if $x \to 5$ using x values increasing to 5, $f(x) \to 1$. So there is no one number L such that $f(x)$ approaches L as x approaches 5. The fact that $f(5)$ makes sense in this example (we defined $f(5) = 1$) is irrelevant. There is no limit as x approaches 5.

Example 6.5.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -2 & \text{if } x \text{ is irrational} \end{cases}.$$

Here $\lim_{x \to 0} f(x)$ does not exist. As x approaches 0, there are always some x values that are rational and some that are irrational. Therefore, there are always $f(x)$ values of 1 and of -2. There is no number L to which the y values become arbitrarily close.

Example 6.6.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ -2 & \text{if } x \text{ is not an integer} \end{cases}.$$

In this case, $\lim_{x\to 0} f(x)$ does exist and is equal to -2. As the x values get very close to 1, there are no integers, so all of the $f(x)$ values are -2. Thus $f(x)$ approaches -2 as x approaches zero. The fact that $f(0) = 1$ is irrelevant. Actually, for this function Here $\lim_{x\to c} f(x) = -2$ for all values of c in $\mathbb{R}$.

In Example 6.4 we discussed x approaching 5 from two directions. There is a notation for this. For Example 6.4 we would write: $\lim_{x\to 5^-} f(x) = 1$ and $\lim_{x\to 5^+} f(x) = -2$. The first is called the *left hand limit* (LHL) and the second is called the *right hand limit* (RHL). For $\lim_{x\to 5} f(x)$ to exist, it must be true that LHL=RHL. Make note that the small $-$ and $+$ superscripts do not indicate that x is positive or negative. They indicate that x is less than 5 or x is greater than 5 respectively.

A very reasonable question at this point is, "So, how do we find limits?" We do NOT find limits by repeatedly substituting in x values closer and closer to see what pattern of y values comes out. The chart for Example 6.1 was an illustration to help with understanding. It isn't how one solves limit problems.

In slower moving calculus courses there would be time for a detailed discussion of limits of sums, differences, products and quotients. Here, we'll go straight to the facts, which result from strict application of mathematical definitions and proof processes. These are some Limit Laws for Finite Limits that you can use to evaluate limits.

Theorem. *If all limits mentioned on each of the following lines exist then*

1. $\lim_{x\to a} c = c$ *for any constant c.*
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
5. $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided* $\lim_{x\to a} g(x) \neq 0$.

By combining parts 1 and 5 of the theorem above we can see that for any constant c we have $\lim_{x\to a} cf(x) = c \cdot (\lim_{x\to a} f(x))$ provided $\lim_{x\to a} f(x)$ exists.

By combining parts 1, 2, 3, 4 and 5 we get the very useful result that $\lim_{x\to a} p(x) = p(a)$ for any polynomial $p(x)$.

Example 6.7. Evaluate the limit: $\lim_{x\to 2}(x^5 - 3x^4 - x^2 + 7)$.

$$\lim_{x\to 2}(x^5 - 3x^4 - x^2 + 7) = 2^5 - 3(2)^4 - (2)^2 + 7 = 32 - 48 - 4 + 7 = -13.$$

There are some other Limit Laws for Finite Limits that require special attention to conditions, particularly to conditions of domain and the existence of some intermediate limits. But, you may use these laws as long as some care is taken to make sure that their usage makes sense.

a. $\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)}$

b. $\lim_{x\to a}[f(x)^n] = [\lim_{x\to a} f(x)]^n$

You'll find we use these facts about limits often.

Example 6.8. Evaluate the limit: $\lim_{x\to 4}\left(\frac{\frac{1}{4}-\frac{1}{x}}{x-4}\right)$

We cannot use Limit Law 6 because the limit in the denominator would be zero. So, we algebraically rewrite our function by combining the fractions in the numerator and simplifying:

$$\lim_{x\to 4}\left(\frac{\frac{1}{4}-\frac{1}{x}}{x-4}\right) = \lim_{x\to 4}\left(\frac{\frac{x-4}{4x}}{\frac{x-4}{1}}\right) = \lim_{x\to 4}\left(\frac{1}{4x}\right) = \frac{1}{16}$$

Example 6.9. Evaluate the limit: $\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

We cannot use Limit Law 6 because the denominator would be zero. So we algebraically rewrite the function, using the conjugate to get rid of the radical.

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} =$$
$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x-1)} = \lim_{x\to 1}(\sqrt{x+3}+2) = \sqrt{1+3}+2 = 4$$

Example 6.10. Find $\lim_{x\to 3} f(x)$, $\lim_{x\to 2} f(x)$ and $\lim_{x\to 0} f(x)$ for: $f(x) = \begin{cases} x+3 & x<2 \\ 2x-1 & 2<x<3 \\ x+2 & x>3 \end{cases}$.

$\lim_{x\to 3^+} f(x) = 3+2 = 5$ and $\lim_{x\to 3^-} f(x) = 2(3)-1 = 5$. Since RHL = LHL, we have $\lim_{x\to 3} f(x) = 5$.
$\lim_{x\to 2^+} f(x) = (2)2-1 = 3$, and $\lim_{x\to 2^-} f(x) = 2+3 = 5$. Since RHL $\neq$ LHL, $\lim_{x\to 2} f(x)$ doesn't exist.
$\lim_{x\to 0} f(x) = 0+3 = 3$. We do not have to use one-sided limits when $x \to 0$ because all values of x very close to 0 are in the interval $x < 2$.

Example 6.11. Find: $\lim\limits_{x\to 2}\dfrac{3x-6}{x^2-4x+4}$. Limit Law 6 doesn't apply so we try a rewrite:

$$\lim_{x\to 2}\frac{3x-6}{x^2-4x+4}=\lim_{x\to 2}\frac{3(x-2)}{(x-2)(x-2)}=\lim_{x\to 2}\frac{1}{x-2}$$

Here we are stuck. This function cannot be simplified further. We still cannot apply Limit Law 6. There is no real number L to which the function values come arbitrarily close. This limit does not exist [14].

Infinite Limits

Let's look back at Example 6.11. We concluded that there was no real number L to which the function values become arbitrarily close. So, by our understanding of "limit," this limit does not exist. But, let's see what *is* happening with the function values as $x \to 2$ for this function.

$x<2$	$\frac{1}{x-2}$	$x>2$	$\frac{1}{x-2}$
1	-1	3	1
1.5	-2	2.5	2
1.9	-10	2.1	10
1.99	-100	2.01	100
1.99999	-100,000	2.00001	100,000
1.9999999	-10,000,000	2.0000001	10,000,000

Look at the values for $x>2$. As the x values get closer and closer to 2, the denominator $(x-2)$ gets closer and closer to zero. So the function itself, the reciprocal of $(x-2)$, gets larger and larger. How large will $\frac{1}{x-2}$ get? Will the function value ever get to be a trillion (12 zeros after the 1)? Yes. From the pattern, you can see that $f(x)$ will be a trillion when $x = 2.000000000001$. Is there a maximum value that $\frac{1}{x-2}$ will attain? No. Do you see that for *any* large number you can pick, the function $\frac{1}{x-2}$ can exceed that number if you choose an x value sufficiently close to 2?

We say that $\lim\limits_{x\to a} f(x)=\infty$ *if* $f(x)$ *becomes unboundedly large as* x *approaches* a.

Our function $f(x)=\dfrac{1}{x-2}$ becomes unboundedly large as x approaches 2 from the right, so we can make the corresponding one-sided limit statement: $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}=\infty$.

Now look at the table values for $x<2$. A similar thing is happening, except that these function values are negative. As $x\to 2$ from the left, the function values are unbounded in the negative direction. We write: $\lim\limits_{x\to 2^-}\dfrac{1}{x-2}=-\infty$.

Since $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}$ is not the same as $\lim\limits_{x\to 2^-}\dfrac{1}{x-2}$, we would say that $\lim\limits_{x\to 2}\dfrac{1}{x-2}$ does not exist.

Most calculus text books define limits to be real numbers, as we did in the first part of this section. Infinity and minus-infinity are not real numbers, so limits like those in Example 6.11 would

[14] There is no FINITE limit for this function. Read the subsection immediately following, entitled "Infinite Limits".

be said to not exist. However, most books then admit that it is convenient to use the limit notation to describe the behavior of functions whose values become unbounded (+ or −) as x approaches some value a. We shall do this too. To merely state that $\lim_{x\to 2} \frac{1}{x-2}$ does not exist is insufficient. One should use the one-sided limit notation to relate the unbounded behavior of the function as $x \to 2$ from each side. More examples will follow, so this should become clear.

The Limit Laws for Finite Limits in the first half of this section do not apply to infinite limits.

So, how do we know we have an infinite limit? Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$ where f and g are functions. If $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$ then we have a situation where the denominator of the fraction is getting very small as x approaches a, but the numerator is not. The value of the function then is becoming unbounded as x approaches a. It is necessary to check the signs of both the numerator and the denominator to see if the unboundness of the quotient is positive or negative.

Example 6.12. Evaluate $\lim_{x\to -3} \frac{x+2}{x+3}$.

Answer: $\lim_{x\to -3}(x+2) = -1$ and $\lim_{x\to -3}(x+3) = 0$. Since the denominator is getting close to zero but the numerator is not, the value of the function is becoming unbounded as x approaches -3.

As x approaches 3 from the left $x < -3$, so the values of $(x+3)$ are negative.

As x approaches 3 from the right $x > -3$, so the values of $(x+3)$ are positive.

As x approaches 3 from either direction, $(x+2)$ approaches -1, which is negative.

So, we conclude: $\lim_{x\to -3^-} \frac{x+2}{x+3} = \infty$ and $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$, so $\lim_{x\to -3} \frac{x+2}{x+3}$ does not exist.

Example 6.13. Evaluate: $\lim_{x\to 0} \frac{1}{x^2}$.

Answer: $\lim_{x\to 0} 1 = 1$, and $\lim_{x\to 0} x^2 = 0$. Since the denominator is getting close to zero but the numerator is not, the value of the function is becoming unbounded as x approaches 0.

As x approaches 0 from either direction, the values of x^2 are positive.

As x apporaches 0 from either direction, the numerator is 1, which is positive.

So, we conclude: $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$ and $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$, so $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 6.14. For $f(x) = \frac{3x+6}{x^2-3x-10}$, find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 5} f(x)$.

Answer: $f(x) = \frac{3x+6}{x^2-3x-10} = \frac{3(x+2)}{(x+2)(x-5)} = \frac{3}{x-5}$ when $x \neq -2$ and $x \neq 5$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{3}{x-5} = -\frac{3}{7}$

$\lim_{x\to 5} 3 = 3$ and $\lim_{x\to 5}(x-5) = 0$, so the value of the function is becoming unbounded as x approaches 5.

As x approaches 5 from the left, $(x-5)$ is negative

As x approaches 5 from the right, $(x-5)$ is positive.

The numerator, 3, is always positive.

So, we conclude $\lim_{x\to 5^-} f(x) = -\infty$ and $\lim_{x\to 5^+} f(x) = \infty$, so $\lim_{x\to 5} f(x)$ does not exist.

Section 6 - Exercises (answers follow)

Find the indicated limit.

1. $\lim_{x\to -2} 6$
2. $\lim_{x\to 1}(3x^2+5x+2)$
3. $\lim_{s\to 0}(2s^3-1)(2s^2+4)$
4. $\lim_{x\to 0}\dfrac{2x-3}{2x-1}$
5. $\lim_{x\to 3}\dfrac{x^2-16}{x-4}$
6. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}$
7. $\lim_{x\to 2}\dfrac{\sqrt{x}-2}{x-4}$
8. $\lim_{x\to 1}\dfrac{[-1/(x+3)]+1/4}{x}$
9. $\lim_{x\to 2} x$
10. $\lim_{x\to 3} e^{2x-1}$
11. $\lim_{x\to 3}\dfrac{x-3}{x-3}$
12. $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$
13. $\lim_{h\to 0}\dfrac{\sqrt{h+9}-3}{h}$
14. $\lim_{y\to 0}\dfrac{6y-9}{y^3-12y+3}$
15. $\lim_{x\to 2}\dfrac{2-x}{\sqrt{7+6x^2}}$
16. $f(x)=\begin{cases}\dfrac{x^2-1}{x+1} & \text{if } x<-1\\ x^2-3 & \text{if } x\geq -1\end{cases}$ Find $\lim_{x\to -1} f(x)$
17. $f(x)=\begin{cases}3+x & \text{if } x<2\\ 3x+1 & \text{if } x\geq 2\end{cases}$ Find $\lim_{x\to 2} f(x)$
18. $f(x)=\begin{cases}3+x & \text{if } x<2\\ 3x-1 & \text{if } x>2\end{cases}$ Find $\lim_{x\to 2} f(x)$
19. $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$
20. $\lim_{z\to 6}\dfrac{z-6}{z^2-36}$
21. $\lim_{z\to 6}\dfrac{z+6}{z^2-36}$
22. $\lim_{x\to 1^-}\dfrac{1+x^2}{1-x^2}$
23. $\lim_{x\to 0^-}\left(\dfrac{1}{x}+\dfrac{1}{x^2-x}\right)$
24. $\lim_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}$
25. $\lim_{u\to 0}\dfrac{\sqrt{u^2+4}-2}{4}$
26. $\lim_{x\to 4}\dfrac{x^2-16}{\sqrt{x+5}-3}$
27. $\lim_{x\to 5}\dfrac{x^2+x-30}{2x-10}$
28. $\lim_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x^2-9}$
29. $\lim_{x\to -1}\dfrac{x^2+3x+2}{x^2+x-6}$
30. $\lim_{x\to 6^-}\dfrac{-4x+3}{x-6}$

31. Consider the graph of $f(x) = \ln x$ (page 37).

(a) What is $\lim\limits_{x\to 0^+} \ln x$? (b) What about $\lim\limits_{x\to 0^-} \ln x$?

32. Find: $\lim\limits_{x\to 0} \frac{|x|}{x}$. Hint: Rewrite the function as a piecewise defined function (see page 13).

33. The statement: $\frac{x^2-9}{x-3} = x+3$ is false, but the statement: $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = \lim\limits_{x\to 3}(x+3)$ is true. Explain.

34. Express the situation described at the beginning of this section (the story about the square drawing machine) as a one-sided limit.

Section 6 - Answers

1. 6 2. 10 3. -4 4. 3 5. 7 6. 5 7. $\frac{1}{2+\sqrt{2}}$ 8. 0 9. 2

10. e^5 11. 1 12. $2x$ 13. $\frac{1}{6}$ 14. -3 15. 0 16. -2

17. $\lim\limits_{x\to 2^-} f(x) = 5$ and $\lim\limits_{x\to 2^+} f(x) = 7$, so $\lim\limits_{x\to 2} f(x)$ does not exist 18. 5 19. 8

20. $\frac{1}{12}$ 21. $\lim\limits_{z\to 6^-} f(x) = -\infty$ and $\lim\limits_{z\to 6^+} f(x) = \infty$, so $\lim\limits_{z\to 6} f(x)$ does not exist 22. ∞

23. -1 24. 6 25. 0 26. 48 27. $\frac{11}{2}$ 28. $-\frac{1}{54}$ 29. 0 30. ∞

31. (a) $-\infty$ (b) This limit makes no sense. There are no values of x less than zero in the domain, so x can't approach from the left.

32. $\lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$ and $\lim\limits_{x\to 0^+} \frac{|x|}{x} = 1$, so $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist.

33. The first statement is not true if $x = 3$. The second statement is a limit where $x \to 3$, so we know that x is not 3.

34. $\lim\limits_{s\to 5^+} s^2 = 25$

7 The Slope of the Tangent to a Graph

We'll start with an example. The graph of $y = x^2 + 1$ is a parabola whose lowest point is $(0, 1)$. The point $(3, 10)$ is on the graph. For any number h, the point $(3 + h, (3 + h)^2 + 1)$ is also on the graph. When $h \neq 0$ this is a different point from $(3, 10)$ because its x-coordinate is different. Assuming $h \neq 0$ we ask:

Question 1. *What is the slope of the line*[15] *joining* $(3, 10)$ *to* $(3 + h, (3 + h)^2 + 1)$*?*

Answer.

$$\frac{[(3+h)^2+1]-10}{(3+h)-3} = \frac{9+6h+h^2+1-10}{h} = \frac{h^2+6h}{h} = h+6$$

Question 2. *Towards what does this slope tend as h approaches 0, and how should we interpret the answer?*

Answer. $h + 6 \to 6$ as $h \to 0$. Interpretation of this lies at the core of calculus:

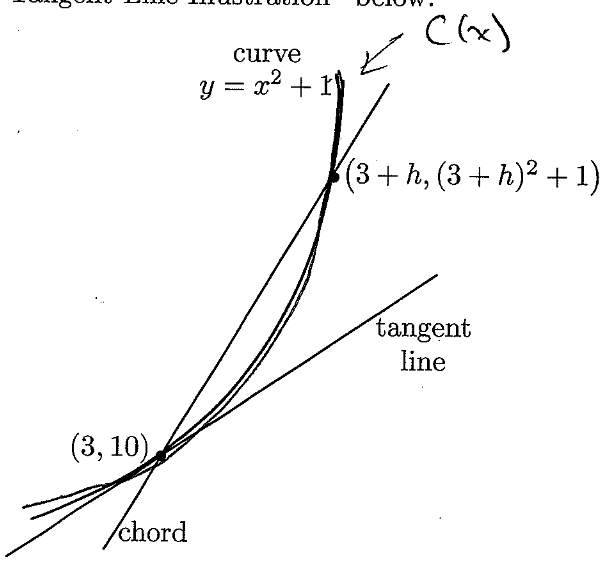
First interpretation (wrong!): When $h = 0$ the point $(3+h, (3+h)^2+1)$ *is* the point $(3, 10)$ so the slope of the line joining $(3, 10)$ to $(3, 10)$ is 6. This is nonsense. There are lots of lines through $(3, 10)$, not just one. Indeed, for any number m the line $y - 10 = m(x - 3)$ has slope m and passes through $(3, 10)$; and the vertical line $x = 3$ also passes through $(3, 10)$.

Second interpretation (right!): The line through $(3, 10)$ with slope 6 is the line

$$y - 10 = 6(x - 3)$$

and this must be a very special line in relation to the graph of $y = x^2 + 1$. We call it the *tangent* to the graph at $(3, 10)$. See "Tangent Line Illustration" below.

Note: Picture is not drawn to scale. The curve $y = x^2 + 1$ is much steeper than the one shown here.



Tangent Line Illustration

[15]The line segment joining two points on a curve is sometimes called a *chord.*

Now let's do the same thing more generally. Consider the function $f(x)$ and, for a moment, let's assume the domain of f is $(-\infty, \infty)$. Pick an x value, say $x = a$. Then $(a, f(a))$ is on the graph of f. For any $h \neq 0$ the slope of the line joining $(a, f(a))$ to the (different) point $(a+h, f(a+h))$ is

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

Question 3. *What happens to this number as $h \to 0$?*

Answer. EITHER: $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ is equal to some number[16], in which case we call that limit $f'(a)$. The line passing through the point $(a, f(a))$ with slope $f'(a)$ is defined to be the *tangent line* to the graph at that point.

OR: $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ is undefined, so there is no number towards which $\frac{f(a+h)-f(a)}{h}$ tends as $h \to 0$, in which case we do not have a tangent line to the graph at the point $(a, f(a))$.

Example 7.1. Find the equation of the tangent line to the curve $f(x) = x^2 + 7x + 1$ at the point $(2, 19)$.

Answer: To find the equation of a tangent line we need a point and a slope. The point given is $(2, 19)$ and the slope is $f'(2)$. We need to find $f'(2)$:

$$\begin{aligned} f(x) &= x^2 + 7x + 1 \\ f(2+h) - f(2) &= [(2+h)^2 + 7(2+h) + 1] - [2^2 + 7 \cdot 2 + 1] \\ &= 4 + 4h + h^2 + 14 + 7h + 1 - 4 - 14 - 1 \\ &= h^2 + 11h \\ \frac{f(2+h)-f(2)}{h} &= h + 11 \end{aligned}$$

So $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}(h+11) = 11$.

The equation of the tangent line, then is $y - 19 = 11(x-2)$.

We now look at three examples where $f'(0)$ does not exist. In the first two examples, there is no tangent line to the graph at $(0, f(0)$. In the third example, there is a line tangent to the graph at $(0, f(0))$, but it is vertical. A sketch of the graphs of these three functions (below) can help you to see the difference.

Example 7.2. Find $f'(0)$ for $f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Answer: Here, $a = 0$.

For $h > 0$, $\frac{f(0+h)-f(0)}{h} = \frac{1-1}{h} = 0$. So, $\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} 0 = 0$

For $h < 0$, $\frac{f(0+h)-f(0)}{h} = \frac{-2-1}{h} = \frac{-3}{h}$. So, $\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{-3}{h} = \infty$

[16] Remember ∞ and $-\infty$ are not numbers.

For $f'(0)$ to exist, the one-sided limits would have to be finite and equal. This is not the case. Also, it doesn't make sense for a slope to be ∞, so certainly $f'(0)$ does not exist.

Example 7.3. Find $f'(0)$ for $f(x) = \sqrt[3]{x}$.

Answer: Here, $a = 0$.

$$\frac{f(0+h) - f(0)}{h} = \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = \frac{\sqrt[3]{h}}{h} = \frac{1}{\sqrt[3]{h^2}}$$

$\lim\limits_{h \to 0^-} \frac{1}{\sqrt[3]{h^2}} = \infty$ and $\lim\limits_{h \to 0^+} \frac{1}{\sqrt[3]{h^2}} = \infty$. So, we can say that $\lim\limits_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = \infty$. However, it makes no sense for a line to have a slope of ∞, so we say that $f'(0)$ does not exist.

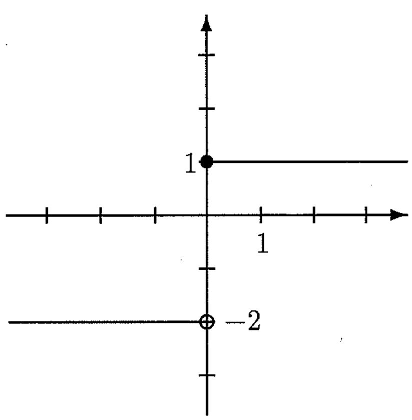
Example 7.4. Find $f'(0)$ for $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Answer: Here, $a = 0$.

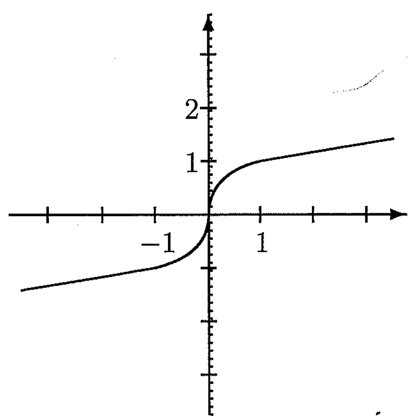
For $h > 0$, $\frac{f(0+h) - f(0)}{h} = \frac{(0+h) - 0}{h} = \frac{h}{h} = 1$ So, $\lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} 1 = 1$

For $h < 0$, $\frac{f(0+h) - f(0)}{h} = \frac{-(0+h) - 0}{h} = \frac{-h}{h} = -1$. So, $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} -1 = -1$.

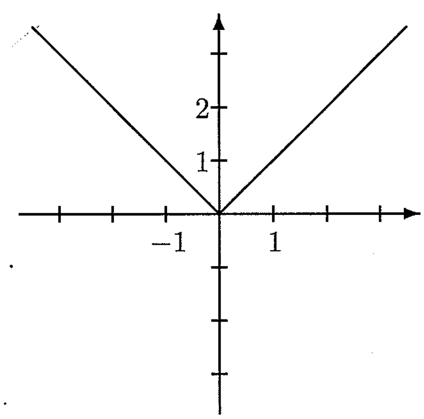
We see that the one-sided limits are not the same so $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$ does not exist.



Example 7.2
No tangent line at $(0, 1)$



Example 7.3
Vertical tangent line at $(0, 0)$



Example 7.4
No tangent line at (0,0)

> We say that $f'(a)$ exists only if $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists and is finite.

Section 7 - Exercises (answers follow)

For exercises 1-3 use the methods of this section. Do not use any short-cut methods that you may have learned previously.

1. Find the *slope* of the tangent line to the graph of each function at the given x value.

   (a) $f(x) = 6$; $x = 2$

   (b) $f(x) = 7 - 5x$; $x = 12$

   (c) $f(x) = \frac{3}{x}$; $x = \frac{1}{2}$

   (d) $f(x) = 2x^2 + x^3$; $x = 5$

   (e) $f(x) = \dfrac{x}{x-1}$; $x = 2$

   (f) $f(x) = \sqrt{x}$; $x = 4$

2. Find the *equation* of the tangent line at the given x values.

   (a) $f(x) = \frac{2}{3x-7}$; $x = 2$

   (b) $f(x) = x^2 - 3x$; $x = -1$

   (c) $f(x) = x + \frac{1}{x}$; $x = 3$

3. $f(x) = \dfrac{1}{x}$

   (a) Find the slope of the line tangent to f at the point $(1, 1)$.

   (b) Look at the graph of f on page 20. At what other point on the graph would you expect the slope of the tangent line to be -1? Check your answer using the appropriate limit.

   (c) Find the slope of the line tangent to f at the point $(4, \frac{1}{4})$.

   (d) Find the slope of the line tangent to f at the point $(\frac{1}{2}, 2)$.

   (e) You have calculated $f'(1)$, $f'(-1)$, $f'(4)$ and $f'(\frac{1}{2})$, the slopes of f at four different points. Did you find the algebra to be repetitive? We can do this in general: Show that for x value a, the slope of the tangent line, $f'(a)$, is $-\frac{1}{a^2}$.

   (f) Check your answers for parts (a) through (d) in the formula for $f'(a)$ given in part (e).

   (g) Observe that $f'(a) = -\dfrac{1}{a^2}$ is always negative. Look again at the graph of $f(x) = \dfrac{1}{x}$. Are there any places on the graph where you would expect the tangent line to have a positive slope?

Section 7 - Answers

1. (a) 0 (b) -5 (c) -12 (d) 95 (e) -1 (f) $\frac{1}{4}$

2. (a) $y + 2 = -6(x - 2)$ (b) $y - 4 = -5(x + 1)$ (c) $y - \frac{10}{3} = \frac{8}{9}(x - 3)$

3. (a) -1 (b) $(-1, -1)$ (c) $-\frac{1}{16}$ (d) -4 (g) No. All tangent lines will slant downward, consistent with a line of negative slope. Any line with positive slope would rise upwards, crossing the graph of f, not touching it tangentially.

8 Derivatives

The Derivative Function

If you have not done Exercise 3 in Section 7, now would be a good time.

In that exercise you learned that you can find an expression for the slope of the tangent line to a function without specifically identifying the point on the function. That is, you could find an expression for $f'(a)$ in terms of a, and then use this expression to find the slope of the tangent line for any specific value of a. We plug un a specific value of a and we get out the slope of the tangent line at $(a, f(a))$. This sounds very much like the behavior of a function. In our discussion so far, a was treated as a constant. It was arbitrary, but constant. Now we will write this limit in function notation, using x as the independent variable. This function has a special name, "derivative."

Definition 8.1. *The derivative of f, denoted f', is the function defined as:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of all numbers x in the domain of f for which this limit exists.

Vocabulary:

1. The process of finding the derivative of f is called *differentiating* f. "To differentiate" is to find the derivative function.

2. In Section 12 we will meet the "derivative of the derivative" which is usually called the "second derivative." So, the derivative introduced here in this section is sometimes called the *first derivative.*

3. If a is in the domain of f and $f'(a)$ exists, we say that f is *differentiable at a.*

4. If I is an open interval lying in the domain of f and if $f'(x)$ exists for all x in I, we say that f is *differentiable on I.*

Example 8.1. For $f(x) = \dfrac{1}{x}$, find $f'(x)$ and the equation of the line tangent to f at the point $(-2, -\frac{1}{2})$.

Answer: From Exercise 3 in Section 7, we get $f'(x) = -\dfrac{1}{x^2}$.

The slope of the tangent line is $f'(-2) = -\dfrac{1}{(-2)^2} = -\dfrac{1}{4}$. So, the equation of the tangent line is $y + \frac{1}{2} = -\frac{1}{4}(x+2)$.

Example 8.2. For $f(x) = \sqrt{x}$, find $f'(x)$, the domain of $f'(x)$, and the slope of the line tangent to f at the point $(4, 2)$.

Answer:

$$\begin{aligned}
f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\
&= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

The domain of f is $[0,\infty)$ but $f'(x)$ does not exist for $x=0$, so the domain of f' is only $(0,\infty)$. The slope of the tangent line at $(4,2)$ is $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

The Derivative is an Instantaneous Rate of Change

Section 7 was about geometry – the slope of the line tangent to a graph at a specific point. Here we interpret the same mathematics quite differently. Look again at the number

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

for some fixed x value a and some number $h \neq 0$. The numerator measures the amount of change (positive or negative or zero), in the value (the y-coordinate) of the function as you move from $x=a$ to $x=(a+h)$. The denominator is the number $(a+h)-a$ and so measures the change in the x-coordinate as you move from a to $a+h$ (a positive change if $h>0$, negative if $h<0$). The above quotient comes from the specific function points $(a, f(a))$ and $(a+h, f(a+h))$. It does not take into account how f behaves at function points between a and $a+h$. We say that $\frac{f(a+h)-f(a)}{h}$ is the *average rate of change* of f between $x=a$ and $x=(a+h)$.

Now consider

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

It is the limit of this average rate of change as $(a+h)$ gets closer and closer to a. We say that this number $f'(a)$ is the *instantaneous rate of change of f at a*. This is an important idea because we are often as interested in the rate at which a function is changing (say, cost or revenue or profit) as we are in the function itself.

The difference between average rate of change and instantaneous rate of change can be thought of this way: Suppose a train is traveling on a track, in one direction. At 3:00 p.m. the train is 10 miles from the station. At 6:00 p.m. the train is 100 miles from the station. The number $\frac{100-10}{6-3} = 30$ tells us that the *average* speed (rate of change of distance compared to time) is 30 mph. This does not tell us anything about the speed of the train at any specific point during those three hours.

In contrast, the *instantaneous rate of change at a point* would give us the speed on the speedometer at a specific instant in time during the three hour trip. That is the value that we would get from the derivative.

YOU NEED TO REMEMBER THAT THE DERIVATIVE MEASURES THE INSTANTANEOUS RATE OF CHANGE. THIS IS A KEY CONCEPT OF CALCULUS.

Recall (from Section 7) that the derivative measures the slope of the tangent line to the graph of f at the point $(a, f(a))$. A large positive derivative suggests a steeply climbing graph, i.e. a fast positive rate of change. A slightly negative derivative suggests a gently falling graph, i.e. a slow negative rate of change[17]. This will be made precise in Section 17.

As x varies, the values of $f'(x)$ tell the whole "rate of change" story of the function f.

Example 8.3. Consider the graph of $f(x) = \dfrac{1}{x}$ on the interval $\frac{1}{3} < x < 3$.

The *average* rate of change of f over this interval is: $\dfrac{f(\frac{1}{3}) - f(3)}{\frac{1}{3} - 3} = \dfrac{3 - \frac{1}{3}}{-\frac{8}{3}} = \dfrac{\frac{8}{3}}{-\frac{8}{3}} = -1$

If you look at the graph of f on page 20 you can see that -1 is a reasonable value for the slope of the line that would go through the points $(\frac{1}{3}, 3)$ and $(3, \frac{1}{3})$. This tells you nothing about the behavior of the graph between these two points.

The *instantaneous* rate of change, the derivative, tells you how the graph is changing at any point in the interval $\frac{1}{3} < x < 3$. We know from Exercise 3 in Section 7 that $f'(x) = -\dfrac{1}{x^2}$. We see that $f'(\frac{1}{2}) = -4$, $f'(\frac{3}{4}) = -\frac{16}{9}$, $f'(1) = -1$, $f'(\frac{3}{2}) = -\frac{4}{9}$, and $f'(2) = -\frac{1}{4}$. The graph is consistent with these derivative values and the idea of a sharply falling graph becoming a more gently falling graph as we increase in x value.

Example 8.4. Rats are infesting City Hall. The Zap-a-Rat company analyzes the situation and claims that they can rid the building of rats within 30 hours. The company shows the mayor the following function: $R(t) = -t^2 + 20t + 290$ where R is the number of rats remaining t hours after the extermination begins. The mayor is impressed by the equation and hires the company. Assuming that Zap-a-Rat's analysis and equation are correct,...

How many rats are currently in City Hall?

Answer: $R(0) = 290$ rats

What is the average rate of change in the quatity of rats from the end of the 5^{th} hour to the end of the 20^{th} hour of the treatment?

Answer: $\dfrac{\Delta R}{\Delta t} = \dfrac{R(20) - R(5)}{20 - 5} = \dfrac{290 - 365}{15} = \dfrac{-75}{15} = -5$ rats/hour.

At what rate is the rat population declining at the end of the 25^{th} hour?

Answer: $R'(t) = -2t + 20$ (verification of this is left as an exercise) $R'(25) = -2(25) + 20 = -30$ rats/hour.

During the extermination process, was the number of rats ever increasing?

[17]Reminder: we always read from left to right and that's how words like "climbing" and "falling" should be understood.

Answer: Yes. $R'(t) > 0$ during the first 10 hours. If the change in the number of rats at any time is positive, it means that the number of rats is increasing at that time.

Will the rats in fact be gone in 30 hours?

Answer: Yes. $R(30) = -(30)^2 + 20(3) + 290 = -10$. In fact, the number of rats is zero when $R(t) = 0$, which is when $t = 10 + \frac{1}{2}\sqrt{1560} \approx 29.75$ hours.

Example 8.5. The cost, in dollars, to produce a product is given as a function of the quantity, q of the product produced: $C(q) = 50,000 + 5q + .01q^2$.

What is the average change in cost if the quantity of product is increased from 100 items to 200 items?

Answer: $\dfrac{\Delta C}{\Delta q} = \dfrac{C(200) - C(100)}{200 - 100} = \dfrac{51,400 - 50,600}{200 - 100} = \dfrac{800}{100} = 8$ dollars/unit.

At what rate is the cost increasing when 120 units are being produced?

Answer: $C'(q) = 5 + .02q$ (again, an exercise). $C'(120) = 5 + .02(120) = \$7.40$ per unit.

In Section 3 we introduced the term *marginal.* It refered to the slope of a linear function. We expand this concept to include the instantaneous rate of change of a function. In Example 8.5 the *marginal cost* is the function $C'(q) = 5 + .02q$. This is consistent also with the interpretation of derivative as slope, as done in Section 7.

Rectilinear (Straight-Line) Motion

The derivative is used in physics for an object that moves in a straight line. Conventionally we think of the path of the object as a horizontal line for side-to-side motion (such as a running person) or a vertical line for up and down motion (such as a rocket shooting skyward and/or falling back to Earth).

The function $s(t)$ gives the position of the object, relative to a fixed point, at time t. We could model our train illustration above: the track is a horizontal line calibrated so that one unit is one mile; the fixed reference point is the station; time is measured in hours past noon. We would then have $s(3) = 10$ and $s(6) = 100$. We could further suppose that at 1:00 p.m. the train is *approaching* the station, and is 20 miles from it. This would give us $s(1) = -20$. What would $s(-2) = -50$ mean? [Answer: It means that at 10:00 a.m. the train is approaching the station and is 50 miles away].

We use the term *velocity* to mean the rate of change of position compared to time. The *average velocity* over time period $t_1 \leq t \leq t_2$ is $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$ and the *instantaneous velocity* at time t is given by the derivative function $s'(t) = v(t)$. Like any average or instantaneous rate of change, velocity can be negative. On a horizontal line, average velocity would be negative if the ending position were to the left of the starting position. On a vertical line, average velocity would be negative if the ending position were lower than the starting position. The instantaneous velocity would be negative if the movement is to the left (or down) and would be positive if the movement is to the right (or up).

While velocity can be negative, *speed* is always positive. Speed is the absolute value of velocity.

Speed $= |v(t)|$. In ordinary life we're more inclined to talk of speed[18] than of velocity ("I drove at 55 mph") but velocity is easier to deal with in math and physics because it isn't an absolute value. Also, it contains directional information which is useful.

Example 8.6. At a carnival shooting range, a target duck moves horizontally for 8 seconds. Its position at time t, measured in centimeters from the center of the target path, is given by the equation $s(t) = t^3 - 3t - 100$.

(a) What is the position of the duck when it begins its motion? What is its position when it ends its motion?
Answers: $s(0) = -100$. The duck is 100 cms. to the left of center.
$s(8) = 388$ The duck is 388 cms. to the right of center.

(b) What is the average velocity of the duck over its entire time of motion?
Answer: $\dfrac{s(8) - s(0)}{8 - 0} = \dfrac{388 - -100}{8} = 61$ cm/sec.

(c) What is the velocity of the duck at $t = 0$? $t = 2$? at $t = 8$?
Answers: $v(t) = s'(t) = 3t^2 - 3$. (This time you can just take my word for it; there are already sufficient exercises for you). $v(0) = -3$. The duck is moving to the left at a speed of 3 cm/sec $v(2) = 3(2)^2 - 3 = 9$ cm/sec. The duck is moving to the right at a speed of 9 cm/sec. $v(8) = 3(8)^2 - 3 = 45$ cm/sec. The duck is moving to the right at the speed of 45 cm/sec.

(d) At what times is the duck moving to the left? right?
Answer: $v(t) = 3t^2 - 3 = 3(t^2 - 1)$ is negative when $t < 1$ and positive when $t > 1$. So, the duck moves to the left for the first second and then moves to the right the rest of the time.

Example 8.7. A ball is shot straight up from the ground with a velocity of 48 ft./sec. Its position above the ground at time t seconds after being launched is given by the equation $s(t) = -16t^2 + 48t$.

(a) When will the ball hit the ground again?
Answer: The ball will be on the ground when $s(t) = 0$. $s(t) = -16t^2 + 48t = -16t(t - 3)$. So, $s(t) = 0$ at $t = 0$ (the initial launch) and $t = 3$ (when it hits the ground again).

(b) What is the average velocity of the ball for the duration of its trip?
Answer: $\dfrac{s(3) - s(0)}{3 - 0} = 0$ ft./sec.

(c) How long was the ball moving upward?
Answer: The ball is moving upward when $v(t) = s'(t) > 0$. $s'(t) = -32t + 48$ (trust me...no exercise). $-32t + 48 = 0$ when $t = \frac{3}{2}$. So the ball rises for 1.5 seconds.

(d) How far does the ball travel all together?
Answer: The distance traveled up is $s(\frac{3}{2}) - s(0) = [-16(\frac{3}{2})^2 + 48(\frac{3}{2})] - 0 = 36$ ft. The distance traveled down is the same as the distance traveled up, so the total distance traveled is 72 ft.

[18]It was correct to use the term "speed" in the original illustration of the train because the train was always traveling in the same positive direction so the velocity was always positive.

Section 8 - Exercises (answers follow)

For all exercises below, compute the derivative of the given function using the method discussed in this section. Do not use short cut formulas you may have learned elsewhere.

1. For each function f, find $f'(x)$

   (a) $f(x) = 4x$

   (b) $f(x) = 6x^2 - 4x$

   (c) $R(t) = -t^2 + 20t + 290$

   (d) $C(q) = 50,000 + 5q + .01q^2$

2. For each function f, find $f'(x)$ and then find $f'(0)$ and $f'(1)$

   (a) $f(x) = x^3 - 2$

   (b) $f(x) = \frac{8}{x}$

   (c) $f(x) = \sqrt{x}$

3. Find the equation of the tangent line to each curve when x has the given value.

   (a) $f(x) = x^2 - 6x^3$; $x = 3$

   (b) $f(x) = 2/x$; $x = 2$

   (c) $f(x) = 11\sqrt{x}$; $x = 5$

4. Suppose the demand (quantity sold) for a certain item is given by $q(p) = -3p^2 + 2p + 1$, where p represents the price of the item in dollars.

   (a) What is the average rate of change in demand when the price is increased from \$7 to \$10?

   (b) Find the rate of change of demand with respect to price.

   (c) Find the rate of change of demand when the price is \$10.

5. An object moves along the x-axis. Its position, in inches relative to the origin, at time t seconds is given by $s(t) = 6t^2 - 4t$. Notice that you have found the derivative for this function in problem 1b above.

   (a) What is the velocity function?

   (b) When is the object moving in the positive direction? negative direction?

   (c) What is the speed of the object at $t = 0$? at $t = 4$?

   (d) What is the total distance traveled (back and forth) by the object between $t = 0$ and $t = 4$?

6. Speed is the absolute value of velocity. Is average speed the absolute value of average velocity? Explain. Hint: Look at Example 8.7.

7. Given $f(x) = x^3 - 5$. Show that the line tangent to the graph of f at the point $(2, 3)$ is parallel to the line tangent to the graph of f at the point $(-2, -13)$.

Section 8 - Answers

1. (a) $f'(x) = 4$
   (b) $f'(x) = 12x - 4$
   (c) $R'(t) = -2t + 20$
   (d) $C'(q) = 5 + .02q$
2. (a) $f'(x) = 3x^2$; $f'(0) = 0$, $f'(1) = 3$
   (b) $f'(x) = \frac{-8}{x^2}$; $f'(0)$ is not defined; $f'(1) = -8$
   (c) $f'(x) = \frac{1}{2\sqrt{x}}$; $f'(0)$ is not defined; $f'(1) = \frac{1}{2}$
3. (a) $y + 153 = -156(x - 3)$
   (b) $y - 1 = -\frac{1}{2}(x - 2)$
   (c) $y - 11\sqrt{5} = \frac{11}{2\sqrt{5}}(x - 5)$
4. (a) -53 items/dollar
   (b) $q'(p) = -6p + 2$
   (c) -58 items/dollar
5. (a) $v(t) = 12t - 4$
   (b) positive direction when $t > \frac{1}{3}$; negative direction when $t < \frac{1}{3}$
   (c) 4 inches/sec 44 inches/sec.
   (d) $81\frac{1}{3}$ inches ($\frac{2}{3}$ inches to the left and then $80\frac{2}{3}$ inches to the right)
6. No. The average speed would only be the absolute value of the average velocity if the velocity was always positive or always negative over the time interval. In the case of Example 8.7, the average velocity is zero. That would only be the average speed if the ball didn't move.
7. Hint: Show that the tangent lines both have the same slope.